DESIGN AND ANALYSIS OF A CLASS OF FUZZY GAIN CONTROLLER

By

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Abstract

In this thesis, a design approach which links up conventional control and fuzzy control methodologies is introduced. The purpose of developing linkage between the two techniques is due to the fact that they show advantages and disadvantages in different aspects. The conventional control is an analytical approach. Its design and analysis is based on mathematical model. Hence, an accurate mathematical model is required for its application. On the other hand, fuzzy control is attractive in its ability to capture human's knowledge or experience into the design, resulting in a controller which is simple to implement. The drawback, however, lies in its lack of a mathematical formulation for design and analysis. Performance of a fuzzy controller is subjective to the designer's expertise. Much efforts may have to be spent on trial and error of various quantities to enhance performance.

The present work introduces performance based fuzzy gain controller to attempt capturing the advantages of the two techniques. The approach introduced

allows a certain degree of freedom in controller design and also includes human-liked control strategies. Moreover, stability may be checked by existing analytical method. In this thesis, stability of a specific class of fuzzy system will be demonstrated.

The present design approach still relies on mathematical models. However, an accurate global linear model is not needed. Instead, localized fuzzy models can be utilized. Linear regression formulation is a well known identification method in conventional system theory. In this thesis, the extended application of linear regression to fuzzy model identification is studied. Furthermore, it will be shown that system membership function can be formulated as weighting factor in an identification process. This leads to the development of fuzzy regional identification technique.

Fuzzy controller exists as a set of database which is operated upon by the heuristic rules in the fuzzy inference system. To tackle the problem of a sizable database, the concept and algorithm of decomposing a fuzzy system into a set of single fuzzy variable subsystems is introduced. The number of fuzzy rules required would be reduced using the subsystem inference. In case the original fuzzy system does not meet the conditions of decomposition, approximation can be carried out via minimization of a quadratic error function.

Contents

Chapter 1

Introduction

1.1 Introduction

The objective of control engineering is to improve performance of a physical system by means of adding controllers. The design of controller can be carried out by empirical methods, thereby various signals are applied to the system and the responses are measured. If the performance of the system is not satisfactory, parameters of the controller are then adjusted. However, such design approach on a trial and error basis is undoubtedly time consuming and non-systematic. Moreover, its application is impractical when experiments are expensive or potentially unstable.

The limitations of empirical methods lead to the development of conventional control theory. Conventional control theory is an analytical approach to the controller design problems. Conventional control theory can be divided into two main classes of studies: *system modeling* and *controller design*. System modeling relates to the determination of a mathematical model for a physical system. A system model can be found by putting together mathematical models of its physical components. In cases of complex or unknown system, the model can be found by identification techniques which generate the mathematical model based upon observed input-output relationship. Once the mathematical model is built, the characteristics of a physical system can be analysed and the controller designed accordingly. In this regard, there exist many design and analysis methods which would guarantee the performance of a controlled system to certain specifications. However, as the conventional approach is based on mathematical models, the main drawback is the requirement of an accurate model. The accuracy of the determined model is usually subjected to experimental uncertainty and environmental noises.

In recent years, fuzzy control emerges as an alternative control design approach. Unlike conventional control design, fuzzy control does not depend on an accurate mathematical model. The controller exists as a set of database operated upon by respective rules in the fuzzy inference system. The database parameters can be designed by summarizing the system operators' knowledge, if available. The advantage of utilizing human knowledge accounts for the ease in implementation of fuzzy controller, especially in task-oriented control applications. Fuzzy control has been successfully applied in various areas such as steam generating unit [1], cement kiln [2], blast furnace [3], and so on. However, problems may arise in case that human knowledge is not available or not satisfactory in performance. The design would have to revert to trial and error basis and hence can be classified as an empirical method. As such, its applicability is limited. Therefore, a systematic way of designing and analyzing fuzzy controller is highly desirable. In addition, as fuzzy control and conventional control show their strength and weakness in different aspects, a linkage between these two techniques would be of importance.

1.2 Review of Previous Work

Fuzzy control was originated by Zadeh [4, 5]. The idea of fuzzy control is to incorporate human experience and knowledge into the controller. The first stage of theory development was mostly contributed by Mamdani [6], then followed by the experimental works of King [7, 2], Ray [1], Yamakawa [8], Zinger [9].

As successful implementation of fuzzy control to numerous practical applications were reported, many investigation on the characteristic of fuzzy control system were also conducted. Some literatures showed that fuzzy system could be represented by explicit formulas (Bouslama [10], Hajjaji [11], Koo [12]), yielding insight of qualitative design and analysis in fuzzy system. Others investigated the dynamic behavior of fuzzy system (Chen [13]) and showed the similarity between fuzzy control and conventional control (Buckley [14], Bouslama [15], Mizumto [16]). Various design approaches were proposed (Huang [17], Han [18], Mizumto [19]), but they all depend on availability of human experience.

The studies of system stability is a very important topic in control engineering and numerous works have been conducted for fuzzy control system. As fuzzy system is not mathematical based, analytical tools developed in conventional control theory cannot be directly applied. Successful stability analysis of fuzzy system via modification of existing tools have been reported (Tanaka [20], Singh [21], Chen [22], Fairnwata [23], Ishigame [24]) on second order systems; however, the analysis for higher order system is very complex.

1.3 Scope of the Thesis

The objective of this study is to search and develop possible linkages between conventional and fuzzy control. The work aims at systematic fuzzy controller design approach by merging the two control design techniques. The outline of the thesis is as follows:

In chapter 2, an introduction to fuzzy control system is given. Definitions of fuzzy set and its basic operations are explained. Two kinds of fuzzy model namely, linguistic model and TSK model, are then derived and briefly compared. Furthermore, the model developed by *Takagi-Sugeno-Kang* (TSK model) which yields insight to combining conventional control and fuzzy control will be discussed. Finally, the basic structure of fuzzy controller will be considered. Detail descriptions of each functional block, such as the fuzzifier, the knowledge base, the inference engine and the defuzzifier, will be presented. Product-Sum-Gravity inference, which will be adopted for the rest of the thesis, will be described.

A fuzzy system can be viewed as a kind of database and there is always problem in handling a large database such as expensive hardware and long search time. In chapter 3, the problem of decomposing a general fuzzy system into a set of subsystems will be studied. Conditions required for the decomposition are also derived. Approximation of a general fuzzy system by decomposed subsystems is demonstrated.

Chapter 4 aims to apply linear regression formulation in fuzzy model identification. The identification can be carried out on-line or off-line. The problem of identification involving different kinds of fuzzy model is studied. It will be shown that membership function can be viewed in the identification process as weighting factor and hence leads to the development of fuzzy regional identification technique.

In chapter 5, performance based fuzzy gain controller and its design algorithm are presented. The design process and non-linearity characteristic of fuzzy system will be analyzed. The stability of a class of fuzzy gain controller is demonstrated.

Lastly, in Chapter 6, a design example is given to illustrate the various concepts of the present work.

Chapter 2

Background Knowledge of Fuzzy Control System

2.1 Introduction

This chapter reviews some background on fuzzy system. First, the basic of fuzzy set theory is introduced in section 2.2. Then, fuzzy models are derived in section 2.3. Finally, the basic structure of a fuzzy controller is shown in section 2.4.

2.2 Fuzzy Sets

Classical sets, or crisp sets, are normally defined as a collection of elements or objects $x \in X$. X is the universe of discourse which may be discrete or continuous. The transition between membership and non-membership in a given set for an element in the universe is abrupt and well-defined. In other words, each element can either belong to or not belong to a set A, where $A \subset X$.

In Fuzzy set theory, this property is generalized [26, 27, 28]. For an element in a universe of fuzzy sets, the transition between membership and non-membership can be gradual. This means an element can partly belong to different fuzzy subsets of the universe of discourse. It is this gradual transition of membership degree that enables fuzzy sets to handle the problems of vague and ambiguous boundaries.

A fuzzy element is hence characterized by its degrees of membership. A fuzzy set $\tilde{A}$ in $\tilde{X}$ may be represented as a set of ordered pairs of a generic element x and its membership degree $\mu_{\tilde{A}}$ as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in \tilde{X}\} \tag{2.1}$$

where,

$$\mu_{\tilde{A}}(x) \in [0, 1]. \tag{2.2}$$

and the membership degree is usually described by a set of discrete values for discrete sets or some functions for continuous sets. The adopted membership functions are usually monotonic, triangular, trapezoidal, and bell-shaped functions as shown in Fig. 2.1(a), (b), (c), and (d), respectively.

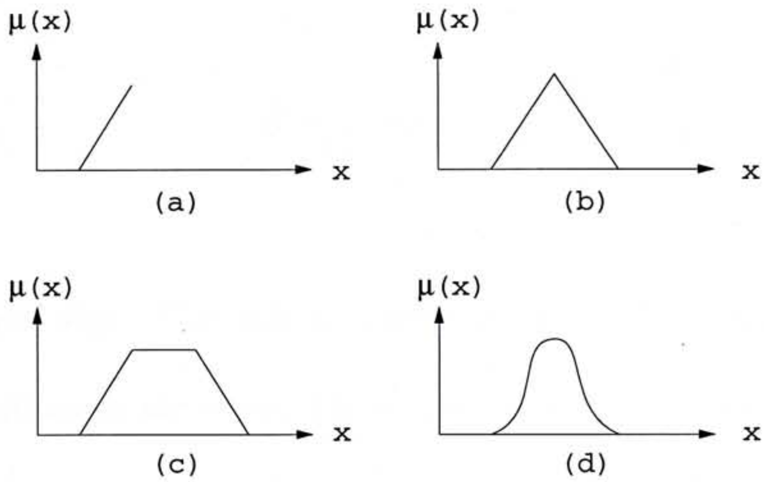


Fig 2.1 (a) Monotonic, (b) Triangular,
(c) Trapezoidal, (d) Bell-shaped functions

When the universe of discourse, $\tilde{X}$, is discrete and finite, a popular notation for fuzzy set $\tilde{A}$ is,

$$\begin{aligned}\tilde{A} &= \mu_{\tilde{A}}(x_1)/x_1 + \cdots + \mu_{\tilde{A}}(x_n)/x_n \\ &= \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)/x_i \end{aligned} \tag{2.3}$$

where the summation symbols "+" and "$\sum$" denote not the algebraic summation but the set theory union operator. Similarly, the oblique line "/" is not a division but the membership degree of an element over a value on the universe of discourse and is not a division. When the universe, $\tilde{X}$, is continuous and infinite, the fuzzy set $\tilde{A}$ is denoted by,

$$\tilde{A} = \int_x \mu_{\tilde{A}}(x)/x. \tag{2.4}$$

where the integral sign "$\int$" is not an algebraic integral but, rather, a set union notation for continuous variables. The oblique line "/" hence assumes the same meaning as in the discrete case.

2.2.1 Properties of Fuzzy Sets

Many of the definitions and properties of fuzzy sets are straightforward extensions of the same from classical set theory. The extended properties usually collapse to their classical counterparts when restricted to fuzzy subsets with membership degrees of a crisp sets, i.e. 0 or 1. On the other hand, fuzzy sets also incorporate definitions and operations which are nonexistent in classical set theory. Some of these are introduced below.

A *normal* fuzzy set is one whose membership function has at least one element in the universe attending unity membership value; otherwise, it is referred as *subnormal.* The largest membership degree of the element within the fuzzy set is called the *height* of a fuzzy set. The membership functions of a normal and subnormal fuzzy set are shown in Fig. 2.2(a), (b).

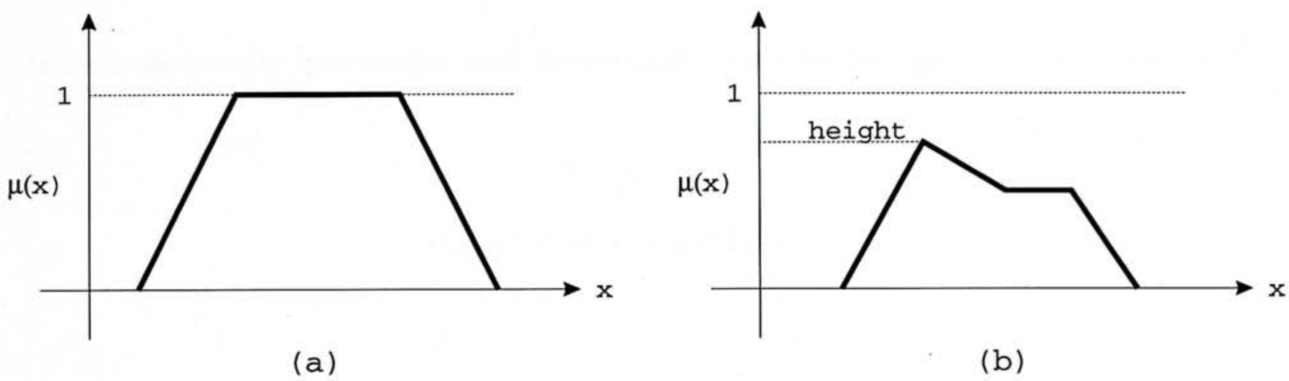


Fig. 2.2 (a) Normal Membership Function
(b) Subnormal Membership Function

The *support* of a fuzzy set $\tilde{A}$ is a crisp set, $S(\tilde{A})$, defined as the region of elements in the universe attending non-zero membership degree. That is, the support is comprised of those elements of the universe, x, where $\mu_{\tilde{A}}(x) \neq 0$.

$$S(\tilde{A}) = \{x \in \tilde{X} | \mu_{\tilde{A}}(x) \geq 0\}. \tag{2.5}$$

The *core* of a fuzzy set $\tilde{A}$ is a crisp set, $Core(A)$, defined as the region of elements in the universe attending unity membership degree. That is, the core is consisting of all elements with membership degree one.

$$Core(A) = \{x \in \tilde{X} | \mu_{\tilde{A}}(x) = 1\}. \tag{2.6}$$

A convex fuzzy set is described by a membership function whose membership values are monotonically increasing or decreasing, or whose membership values are monotonically increasing and then monotonically decreasing. In other words, if

$$\mu_{\tilde{A}}(y) \geq \min[\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(z)] \tag{2.7}$$

for all elements in a fuzzy set $\tilde{A}$ with $x < y < z$,then $\tilde{A}$ is said to be a convex fuzzy set. Fig. 2.3 shows a convex and a non-convex set.

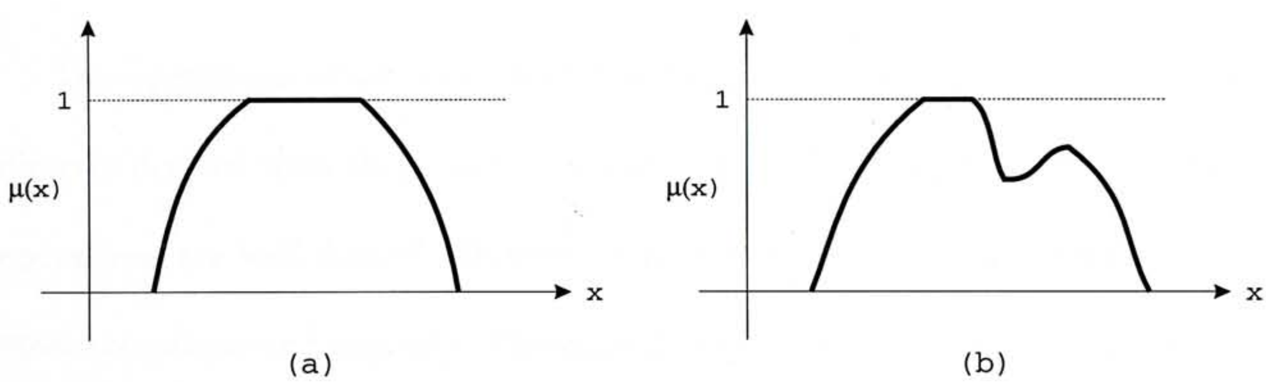


Fig. 2.3 (a) Convex Set; (b) Non-convex Set

2.2.2 Operations on Fuzzy Sets

Definitions of *equality* and *inclusion* in classical set theory can be straightforwardly extended to fuzzy set making use of the membership functions. Two fuzzy sets $\tilde{A}$ and $\tilde{B}$ are *equal* if and only if every element in $\tilde{A}$ has the same membership degree as it has in $\tilde{B}$. i.e.

$$\tilde{A} = \tilde{B} \quad \Leftrightarrow \quad \forall x \in \tilde{X} : \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x). \tag{2.8}$$

Similarly, a fuzzy set $\tilde{A}$ is said to be a *subset* of a fuzzy set $\tilde{B}$ if and only if every element in $\tilde{A}$ has a lower membership degree than it has in $\tilde{B}$. i.e.

$$\tilde{A} \subseteq \tilde{B} \quad \Leftrightarrow \quad \forall x \in \tilde{X} : \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x). \tag{2.9}$$

The operations *union, intersection* and *complement* of fuzzy sets cannot be directly derived from their classical counterparts. In classical set theory, these operations are well defined. However, the interpretation of these operations is more complicate in fuzzy sets. There are actually more than one possible way to give a consistent definition. The one proposed by Zadeh [4] is the most widely accepted:

$$\textit{union} \qquad \forall x \in \tilde{X} \ : \mu_{\tilde{A}\cup\tilde{B}}(x) = \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \tag{2.10}$$

$$\textit{intersection} \qquad \forall x \in \tilde{X} \ : \mu_{\tilde{A} \cap \tilde{B}}(x) = \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \tag{2.11}$$

$$\textit{complement} \qquad \forall x \in \tilde{X} \ : \mu_{\tilde{A}'}(x) = 1 - \mu_{\tilde{A}}(x). \tag{2.12}$$

The Venn diagrams for these operations, extended to consider fuzzy sets, are shown in Fig. 2.4.

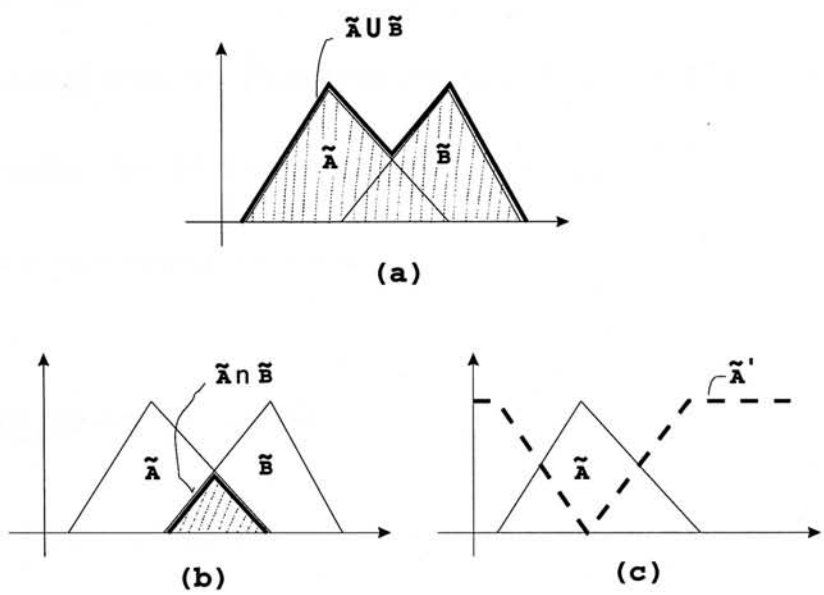


Fig. 2.4 Venn Diagrams for (a) Union; (b) Intersection; (c) Complement

2.3 Fuzzy Models

Fuzzy systems models basically fall into two categories representing different types of information. The first type is Linguistic Model which is based on collections of *IF - THEN* rules with vague predicates. Here, fuzzy variables are associated with linguistic terms. This type of fuzzy models utilizes fuzzy reasoning [5, 29, 30] and form the basis of qualitative modeling which describes system behavior by natural language [28, 31, 32].

The other type of fuzzy models is based on the work of *Takagi-Sugeno-Kang* (TSK) [33, 34]. These models are formed by logical rules that have a fuzzy antecedent part and a functional consequent part. They are a combination of fuzzy and non-fuzzy models. Fuzzy models based on the TSK model exhibit the ability of Linguistic Model for qualitative knowledge representation as well as the potential for expressing quantitative information.

2.3.1 Linguistic Model

The main directions in linguistic fuzzy systems theory is originally initiated by Zadeh [5]. This approach describes models by means of inference rules which are based on verbs. The rule-set replaces the usual set of equations for characterization of a system.

The Linguistic Model is a knowledge-based system. It contains rules which incorporate inherent fuzzy knowledge. The primary concept of Linguistic Model is that the input space is partitioned into fuzzy regions, each of which has with its own associated output. The role of the fuzzy sets is to form mappings of input-output values according to individual rules. The general expression of Linguistic Model is:

$$
\begin{array}{llllllll}
\text{Rule 1:} & \text{IF } x_1 \text{ is } X_{1,1} \textit{ and} & \ldots & \textit{and} & x_n \text{ is } X_{n,1} & \Rightarrow & u \text{ is } U_1 \\
 & \vdots & \vdots & & \vdots & & \vdots \\
\text{Rule m:} & \text{IF } x_1 \text{ is } X_{1,m} \textit{ and} & \ldots & \textit{and} & x_n \text{ is } X_{n,m} & \Rightarrow & u \text{ is } U_m
\end{array}
\tag{2.13}
$$

where m is the total number of rules, $(x_1, \ldots, x_n)$ and u are the input and output variables, respectively. $(X_{1,i}, \ldots, X_{n,i})$ and U_i are the corresponding linguistic terms which are usually expressed as *small, medium, high* and so on. The linguistic terms may quantitatively be expressed by single value.

2.3.2 Takagi-Sugeno-Kang (TSK) Fuzzy Model

Linguistic model suffers from its inability of incorporating mathematical formula into the fuzzy set frame-work whereas this kind of knowledge regarding the physical systems is often available. To overcome this, Sugeno and co-workers proposed an alternative fuzzy reasoning method [33, 34] capable of incorporating mathematical formula into fuzzy inference. The TSK model has basically the same structure as that of the linguistic model. The input space of both are defined by linguistic variables. The difference is that the output consequence in the TSK model is expressed as a function rather than as a linguistic variable. The general expression of TSK model is:

$$\begin{array}{lllll} \text{Rule 1:} & \text{IF } x_1 \text{ is } X_{1,1} \textit{ and} & \ldots & \textit{and } x_n \text{ is } X_{n,1} & \Rightarrow \ u_1 = f_1(x_1,\ldots,x_n) \\ & \vdots & \vdots & \vdots & \vdots \\ \text{Rule m:} & \text{IF } x_1 \text{ is } X_{1,m} \textit{ and} & \ldots & \textit{and } x_n \text{ is } X_{n,m} & \Rightarrow \ u_m = f_m(x_1,\ldots,x_n) \end{array} \tag{2.14}$$

where f_i, $i = 1, \ldots, m$ are now functions of the input variables $(x_1, \ldots, x_n)$. It should be noted that, if the function f_i is defined in the form, $f_i = a_{o,i} + a_{1,i}x_1 + \ldots + a_{n,i}x_n$, then each fuzzy rule in eqn. (2.14) adopts a particular linear model as output. As such, traditional system design and analysis methods may be applied. It should also be noted that, the linguistic model can be regarded as a special case of the TSK model when $a_{k,i} = 0$ for $k = 1, \ldots, n$.

2.4 Fuzzy Inference System

A simple structure of a fuzzy inference system is shown in Fig. 2.5. The architecture is comprised of four modules. They are the fuzzifier, knowledge base, inference engine, and defuzzifier. The functions of each module are described in the following.

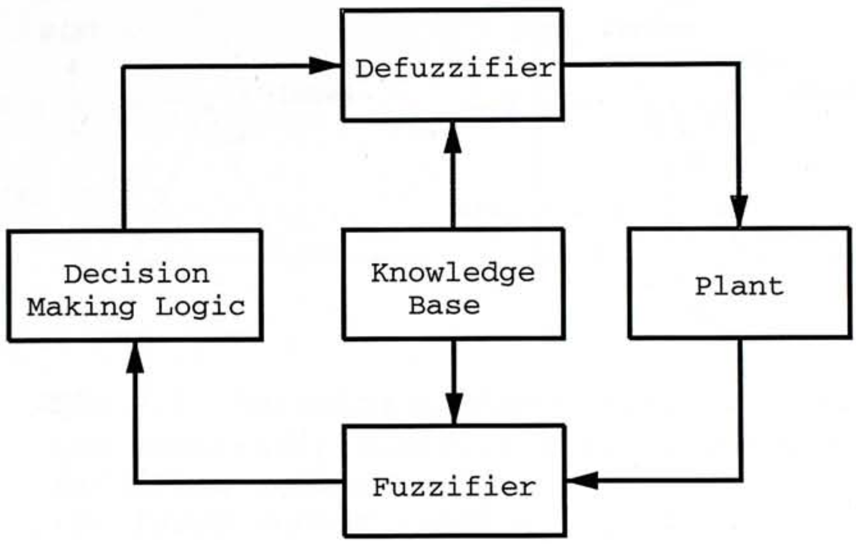


Fig 2.5 A Simple Structure of Fuzzy Inference System

2.4.1 Fuzzifier

A fuzzifier is a coder which codes the senor measurements in terms of the linguistic labels in the fuzzy rules. If the sensor reading has a crisp value, then the fuzzification stage amounts to matching the sensor measurement against the membership function of the linguistic label as shown in Fig 2.6(a). If the sensor reading contains noise, it may be modeled as a triangular function. The vertex of the triangle, which refers to the mean value of the data set, is then used in fuzzification stage as shown in Fig. 2.6(b). The former case is more widely used as sensor reading can generally be considered as crisp measurement.

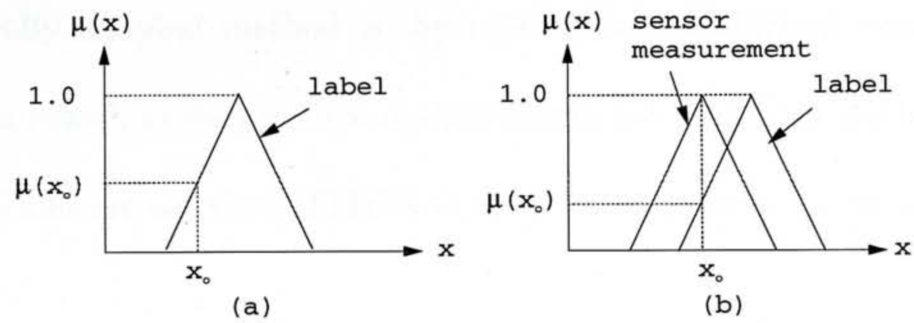


Fig 2.6 Matching a sensor reading x_o with the membership function $\mu(x)$ to get $\mu(x_o)$;
(a) crisp sensor reading,
(b) fuzzy sensor reading.

2.4.2 Knowledge Base

The knowledge base contains the available system information within the domain of interest. In reality, this knowledge is acquired either directly from human experience or indirectly from conducting experiments on the system. The main challenge of setting up the knowledge base is the selection of a set of linguistic variables or a set of equations for TSK model which would properly describes the output of the system.

2.4.3 Inference Engine

The inference engine is a mechanism for manipulating the rules in the knowledge base to form inferences and to draw conclusion. The conclusion can be deduced in a number of ways depending on the structure of the engine. The most generally adopted method is the rules-driven production systems. The "production rules", as they are commonly called, are usually in the form "**IF** (a set of conditions are satisfied) **THEN** (a set of consequences can be produced)".

2.4.4 Defuzzifier

The defuzzifier converts the fuzzy consequences into a non-fuzzy values. The most commonly used defuzzification techniques are the Center of Gravity (COG) method and the Mean of Maximum (MOM) method.

In COG method, the output is defuzzified as

$$u = \frac{\int \sum_{i=1}^{m} u_o * \mu_{U_i} * U_i(u)du}{\int \sum_{i=1}^{m} \mu_{U_i} * U_i(u)du} \tag{2.15}$$

for continuous universe, and as

$$u = \frac{\sum\limits_{i=1}^{m} u_i * \mu_{U_i}}{\sum\limits_{i=1}^{m} \mu_{U_i}} \tag{2.16}$$

for discrete universe, where in eqn. (2.15) and (2.16), m is the number of fuzzy rules, u_i is the output for rule i, $U_i(u)$ is the membership function and μ_{U_i} is the membership value for rule i.

In MOM method, the output is defuzzified as

$$u = \sum_{i=1}^{n} \frac{u_i(max(\mu_{U_i}))}{n} \tag{2.17}$$

where n is the number of fuzzy rules attaining maximum membership. If there is only one rule has attaining maximum membership, $n = 1$.

2.4.5 Product-Sum-Gravity Inference

This section introduces the **Product-Sum-Gravity Inference** which will be adopted throughout the remainder of the thesis. For better illustration of concepts, the inference is first introduced using a system with 2 inputs each of which has 3 membership functions.

Consider an inference system operating on fuzzy variables x and y with fuzzy rules:

Rule 1:	If	x is X_1	and	y is Y_1	$\Rightarrow$	u is $U_{1,1}$
Rule 2:	If	x is X_1	and	y is Y_2	$\Rightarrow$	u is $U_{1,2}$
Rule 3:	If	x is X_1	and	y is Y_3	$\Rightarrow$	u is $U_{1,3}$
Rule 4:	If	x is X_2	and	y is Y_1	$\Rightarrow$	u is $U_{2,1}$

$$\begin{array}{llllll} \text{Rule 5:} & \text{If} \;\; x \text{ is } X_2 & \text{and} \;\; y \text{ is } Y_2 & \Rightarrow & u \text{ is } U_{2,2} \\ \text{Rule 6:} & \text{If} \;\; x \text{ is } X_2 & \text{and} \;\; y \text{ is } Y_3 & \Rightarrow & u \text{ is } U_{2,3} \\ \text{Rule 7:} & \text{If} \;\; x \text{ is } X_3 & \text{and} \;\; y \text{ is } Y_1 & \Rightarrow & u \text{ is } U_{3,1} \\ \text{Rule 8:} & \text{If} \;\; x \text{ is } X_3 & \text{and} \;\; y \text{ is } Y_2 & \Rightarrow & u \text{ is } U_{3,2} \\ \text{Rule 9:} & \text{If} \;\; x \text{ is } X_3 & \text{and} \;\; y \text{ is } Y_3 & \Rightarrow & u \text{ is } U_{3,3} \end{array} \quad (2.18)$$

where $X_i(x)$ and $Y_j(y)$, $i, j = 1, 2, 3$, are the antecedent membership functions, and $U_{i,j}(u)$, $i, j = 1, 2, 3$, are the consequent membership functions. Given the fact $x = x_o$, $y = y_o$, the representative value $u = u_o$ is inferred using the following:

Product: For each rule, the membership value is the product of antecedent membership degrees of x_o and y_o. The 4th rule in eqn. (2.18), for example, has a membership value

$$\mu_{U_{2,1}} = \mu_{X_2}(x_o) * \mu_{Y_1}(y_o) \quad (2.19)$$

where $\mu_{X_i}(x_o)$ and $\mu_{Y_j}(y_o)$ are degrees of memberships of x_o and y_o with respect to $X_i(x)$ and $Y_j(y)$.

Sum: Distribution of the overall consequence $U(u)$ is the sum of $\mu_{U_{i,j}} * U_{i,j}(u)$ over all rules,

$$U(u) = \sum_{i=1}^{3} \sum_{j=1}^{3} \mu_{U_{i,j}} * U_{i,j} \tag{2.20}$$

Gravity: The representative output u_o is obtained as the center of gravity of $U(u)$,

$$\begin{aligned} u_o &= \frac{\int u * U(u)du}{\int U(u)du} \\ &= \frac{\sum_{i=1}^{3} \sum_{j=1}^{3} \int u * \mu_{U_{i,j}} * U_{i,j}(u)du}{\sum_{i=1}^{3} \sum_{j=1}^{3} \int \mu_{U_{i,j}} * U_{i,j}(u)du} \end{aligned} \tag{2.21}$$

Eqn. (2.21) can be simplified assuming a symmetric function is adopted for all $U_{i,j}(u)$ as shown in Fig. 2.7.

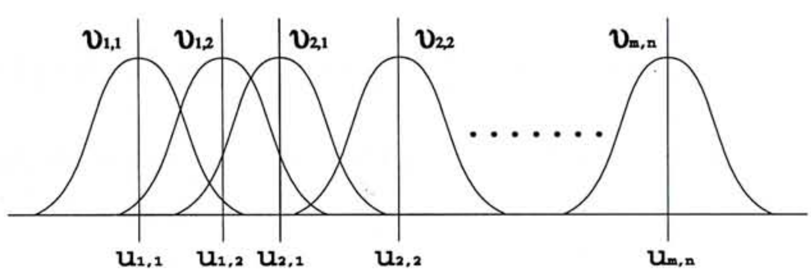


Fig 2.7 Consequent Membership Functions

$$u_o = \frac{\sum_{i=1}^{3} \sum_{j=1}^{3} u_{i,j} * \mu_{U_{i,j}}}{\sum_{i=1}^{3} \sum_{j=1}^{3} \mu_{U_{i,j}}} \tag{2.22}$$

where $u_{i,j}$ denote locations of the axes of symmetry of $U_{i,j}$ for $i, j = 1, 2, 3$. Eqn.

(2.22) implies that, in this case, the consequent membership function $U_{i,j}$ can actually be replaced by $\delta(u_{i,j})$, the singleton function at $u_{i,j}$. The general term of the system can be rewritten as

$$\text{Rule ij:} \quad \text{If} \quad x \text{ is } X_i \quad \text{and} \quad y \text{ is } Y_j \quad \Rightarrow \quad u \text{ is } \delta(u_{i,j})$$

For a general system with n fuzzy variables, $x_1, x_2, \ldots, x_n$, each with membership functions $X_{1,i}, X_{2,j}, \ldots, X_{n,k}$, $i, j, \ldots, k = 1, \ldots, m$, there are m^n fuzzy rules and m^n consequent membership functions $U_{i,j,\ldots,k}$ with axes of symmetry at $u_{i,j,\ldots,k}$. The fuzzy rules are

$$\text{Rule N:} \quad \text{If } x_1 \text{ is } X_{1,i}\text{, } x_2 \text{ is } X_{2,j}\text{, } \cdots\text{, } x_n \text{ is } X_{n,k} \;\Rightarrow\; u \text{ is } U_{i,j,\ldots,k} \tag{2.23}$$

with $N = [(i-1) * m^{n-1} + (j-1) * m^{n-2} + \cdots + k]$ for $i, j, \ldots, k = 1, 2, \ldots, m$. Given $x_1 = x_{1,(o)}, x_2 = x_{2,(o)}, \ldots, x_n = x_{n,(o)}$, the Product-Sum-Gravity inference yields

$$\mu_{U_{i,j,\ldots,k}} = \mu_{X_{1,i}}(x_{1,(o)}) * \mu_{X_{2,j}}(x_{2,(o)}) * \cdots * \mu_{X_{n,k}}(x_{n,(o)})$$

and

$$u_o = \frac{\sum_{i=1}^{m} \sum_{j=1}^{m} \cdots \sum_{k=1}^{m} u_{i,j,\ldots,k} * \mu_{U_{i,j,\ldots,k}}}{\sum_{i=1}^{m} \sum_{j=1}^{m} \cdots \sum_{k=1}^{m} \mu_{U_{i,j,\ldots,k}}} \tag{2.24}$$

Chapter 3

Decomposition of Fuzzy Rules

3.1 Introduction

Fuzzy logic system can be viewed as a kind of database whereby the data are interpolated by means of inference mechanism [28, 26]. As the number of linguistic terms for a variable increase, the required database would expand geometrically. As such, fuzzy system is susceptible to problems faced by large database such as storage space and search time. Attempt to deal with this problem by means of condensed look-up table has been reported [17]. In this chapter, a decomposition method to reduce the number of fuzzy rules by subsystem inference [35] will be presented. For better illustration of concepts, results is again derived first using a fuzzy system with 2 inputs and each of which has 3 membership functions. The general case is then deduced.

3.2 Decomposability of Fuzzy Inference System

Consider the following system

$$\text{Rule } ij: \quad \text{If } X_i \text{ and } Y_j \quad \Rightarrow \quad \delta(u_{ij}) \tag{3.1}$$

where, $i = 1, \ldots, 3$ and $j = 1, \ldots, 3$ and $\delta(u_{ij})$ is the singleton function at u_{ij}. The result here would hold if a symmetric function is adopted for consequent membership functions.

With "Product-Sum-Gravity" inference, the output u_o is obtained as (see equation 2.22),

$$u_o = \frac{\sum\limits_{i=1}^{3} \sum\limits_{j=1}^{3} u_{i,j} * \mu_{u_{X,i}} * \mu_{u_{Y,j}}}{\sum\limits_{i=1}^{3} \sum\limits_{j=1}^{3} \mu_{u_{X,i}} * \mu_{u_{Y,j}}} \tag{3.2}$$

If there exists constants, α_i and β_j, satisfying:

$$u_{i,j} = \alpha_i + \beta_j. \tag{3.3}$$

one has, substituting eqn. (3.3) into eqn. (3.2),

$$\begin{aligned}
u_o &= \frac{\sum_{i=1}^{3}\sum_{j=1}^{3}(\alpha_i+\beta_j)*\mu_{X_i}*\mu_{Y_j}}{\sum_{i=1}^{3}\sum_{j=1}^{3}\mu_{X_i}*\mu_{Y_j}} && (3.4)\\
&= \frac{\sum_{i=1}^{3}\sum_{j=1}^{3}\alpha_i*\mu_{X_i}*\mu_{Y_j}}{\sum_{i=1}^{3}\sum_{j=1}^{3}\mu_{X_i}*\mu_{Y_j}} + \frac{\sum_{i=1}^{3}\sum_{j=1}^{3}\beta_j*\mu_{X_i}*\mu_{Y_j}}{\sum_{i=1}^{3}\sum_{j=1}^{3}\mu_{X_i}*\mu_{Y_j}} && (3.5)\\
&= \frac{\sum_{i=1}^{3}\alpha_i*\mu_{X_i}}{\sum_{i=1}^{3}\mu_{X_i}} + \frac{\sum_{j=1}^{3}\beta_j*\mu_{Y_j}}{\sum_{j=1}^{3}\mu_{Y_j}} && (3.6)\\
&= u_{X,o} + u_{Y,o} && (3.7)
\end{aligned}$$

Outputs $u_{X,o}$ and $u_{Y,o}$ are, respectively, the "Product-Sum-Gravity" inferred results from inference subsystem X:

$$\text{Rule } i \ : \text{If } X_i \ \Rightarrow \ \delta(\alpha_i), \quad \text{for } i = 1, \ldots, 3 \tag{3.8}$$

and subsystem Y:

$$\text{Rule } j \;:\; \text{If } Y_j \;\Rightarrow\; \delta(\beta_j), \quad \text{for } j = 1, \ldots, 3 \tag{3.9}$$

where $\delta(\alpha_i)$ and $\delta(\beta_j)$ denoted the singleton function at α_i and β_j, respectively.

The above shows that eqn. (3.3) is a sufficient condition that the original inference system can be "decomposed" into two inference subsystems X and Y operating on respective fuzzy variables of x and y. On the reverse, if it is given that $u_o = u_{X,o} + u_{Y,o}$ where $u_{X,o}$ and $u_{Y,o}$ are inferred outputs of subsystem X and Y, as given in (3.8) and (3.9), one can easily work backward from eqn. (3.7) to (3.4) to arrive at the conclusion of $u_{ij} = \alpha_i + \beta_j$. Eqn. (3.3) is hence the necessary and sufficient condition that fuzzy system (3.1) is decomposable into two subsystem operating on single fuzzy variable.

In the general case, where there are n fuzzy variables $x_1, x_2, \ldots, x_n$, each with m linguistic terms, condition for "decomposability" becomes: existence of $m * n$ constants $\alpha_i, \beta_j, \ldots, \xi_k$ where $i, j, \ldots, k = 1, \ldots, m$, such that

$$u_{i,j,\ldots,k} = \alpha_i + \beta_j + \cdots + \xi_k \tag{3.10}$$

The system output u_o is then subsequently given by $u_o = u_{X_{1,o}} + u_{X_{2,o}} + \cdots +$

$u_{X_{n,o}}$ where $u_{X_{1,o}}, u_{X_{2,o}}, \ldots, u_{X_{n,o}}$ are each generated base on the "Product-Sum-Gravity" inference subsystems $X_1, X_2, \ldots, X_n$ given as

$$
\begin{array}{ll}
\textit{Subsystem} \;\; X_1 : & \text{If } X_{1,i} \Rightarrow \delta(\alpha_i), \text{ for } i = 1, \ldots, m \\
\textit{Subsystem} \;\; X_2 : & \text{If } X_{2,j} \Rightarrow \delta(\beta_j), \text{ for } j = 1, \ldots, m \\
\vdots & \qquad\qquad \vdots \\
\textit{Subsystem} \;\; X_n : & \text{If } X_{n,k} \Rightarrow \delta(\xi_k), \text{ for } k = 1, \ldots, m
\end{array}
$$

Note that the original system has m^n fuzzy rules while the subsystems together has $m * n$ rules. A decomposed system hence requires a reduced number of fuzzy rules and a reduced database.

3.3 The Decomposability condition

The decomposability condition depends only on the positioning of the consequent membership functions. Consider again system (3.1) for illustration. The decomposability condition, eqn. (3.3) can be written in matrix form:

$$u = \mathcal{D}p \tag{3.11}$$

where,

$$u = \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ u_{3,1} \\ u_{3,2} \\ u_{3,3} \end{bmatrix}, \mathcal{D} = \begin{bmatrix} 1\,0\,0\,1\,0\,0 \\ 1\,0\,0\,0\,1\,0 \\ 1\,0\,0\,0\,0\,1 \\ 0\,1\,0\,1\,0\,0 \\ 0\,1\,0\,0\,1\,0 \\ 0\,1\,0\,0\,0\,1 \\ 0\,0\,1\,1\,0\,0 \\ 0\,0\,1\,0\,1\,0 \\ 0\,0\,1\,0\,0\,1 \end{bmatrix}, p = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}.$$

The matrix $\mathcal{D}$ is of rank 5. This can be observed from the fact that the sum of the first three columns of $\mathcal{D}$ (columns multiply α_i) equals the sum of the second three columns (columns multiply β_i), both of which equal the column vector of all 1's. Therefore, only five out of six columns are linearly independent. Hence, the orthogonal space of column vectors of $\mathcal{D}$, has 4 independent basis vectors. If one forms a 9 by 4 matrix $\mathcal{D}_\perp$ by gathering any four linear independent basis vectors of the orthogonal space and multiplies its transpose to eqn. (3.11), one has

$$\mathcal{D}_\perp^T u = 0 \tag{3.12}$$

as $\mathcal{D}_\perp^T\mathcal{D} = 0$. Eqn. (3.12) corresponds to 4 equations with 9 unknowns. Hence, to achieve decomposability, not all $u_{i,j}$ can be chosen arbitrary. Only 5 of the 9 $u_{i,j}$s are independent variables, the remaining 4 would then follow as dictated by eqn. (3.12).

The principle of selecting $u_{i,j}$ to satisfy eqn. (3.12) can be shown by putting $u_{i,j}$s in a matrix form:

$$\mathcal{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix}$$

With $u_{i,j} = \alpha_i + \beta_j$, it can be seen that the difference between the first and second row of $\mathcal{U}$ equals $(\alpha_1 - \alpha_2) * [1\ 1\ 1]$ and that the difference between the second and third row equals $(\alpha_2 - \alpha_3) * [1\ 1\ 1]$. Likewise, the difference equals $(\beta_1 - \beta_2) * [1\ 1\ 1]^T$ and $(\beta_2 - \beta_3) * [1\ 1\ 1]^T$, respectively, between the first and second column, and the second and third column. In other word, the difference of the elements between the ith and jth row of $\mathcal{U}$ is a constant which depends on α_i and α_j. Similar property holds for the columns with the constant depending on β_i and β_j. That means the matrix $\mathcal{U}$ forms "Lattice" structure. The selection of independent variables for $u_{i,j}$s should be guided by this "Lattice" property.

For example, one can pick the set of $u_{1,1}, u_{1,2}, u_{1,3}, u_{2,1}$, and $u_{3,1}$ as independent variables, or $u_{1,1}, u_{1,2}, u_{2,1}, u_{2,3}$, and $u_{3,3}$, but not $u_{1,1}, u_{1,2}, u_{2,1}, u_{2,2}$, and $u_{3,3}$, as the first four members of the latest set are related by the "Lattice" property and are not linear independent of each other. After picking the independent variables, the remaining $u_{i,j}$s are deduced accordingly.

The same reasoning follows for the general case of m^n rules. The corresponding matrix $\mathcal{D}$ for equation (3.11) has dimension of m^n by $m * n$. Noting that the sum of the first m columns of $\mathcal{D}$ equals to that of the second m columns (and equals to a vector with all 1's), one of the second m columns can hence be regarded as linear dependent on others. Again, with the same argument applied to the third m columns of $\mathcal{D}$, and so on, it can be concluded that the rank of $\mathcal{D}$ equals $m * n - (n - 1)$, **(total no. of rules - (no. of fuzzy inputs - 1))**. If the output $u_{i,j,\ldots,k}$ forms a n dimensional hypercube matrix $\mathcal{U}$, the "Lattice" structure maintains. Therefore, $m*n-n+1$ independent $u_{i,j,\ldots,k}$s can be selected and the remaining $u_{i,j,\ldots,k}$s can be deduced according to the "Lattice" property.

3.4 Determining Decomposed Parameters

If the value $u_{i,j}$s satisfy the "Lattice" property, parameters α_i and β_j where $i, j = 1, 2, 3$ can be determined from equation (3.11). As there are 5 linear independent columns of $\mathcal{D}$, the 6th column of $\mathcal{D}$ can be considered as the depending column and (3.11) can be re-written as

$$u = \bar{\mathcal{D}}\bar{p}$$

where $\bar{\mathcal{D}}$ is the first 5 columns of $\mathcal{D}$, $\bar{p} = \mathcal{C}p$ is a reduced parameter vector, and

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

With $\bar{\mathcal{D}}^{\dagger} = (\bar{\mathcal{D}}^T\bar{\mathcal{D}})^{-1}\bar{\mathcal{D}}^T$ as pseudo-inverse of $\bar{\mathcal{D}}$

$$\bar{p} = \bar{\mathcal{D}}^{\dagger}u \tag{3.13}$$

Parameter p can be determined from $\bar{p}$ as

$$p = \bar{\mathcal{D}}^{\dagger}u - \beta_3 * [1\ 1\ 1\ \text{-}1\ \text{-}1]^T$$

Parameter β_3 can be arbitrary chosen without affecting the fact $u_{i,j} = \alpha_i + \beta_j$. A convenient chosen could be β_3 as zero. Once β_3 is selected, p can be obtained.

For the general case of n fuzzy variables and each with m linguistic terms, $\mathcal{D}$ in eqn. (3.12) has dimension m^n by $m*n$ and rank $m*n-(n-1)$. The matrix $\bar{\mathcal{D}}$ of dimension m^n by $(m*n-(n-1))$ can be formed by excluding the $2m$th, $3m$th, ..., and $n*m$th columns of matrix $\mathcal{D}$. Matrix $\bar{\mathcal{D}}$ is of rank $m*n-(n-1)$. Correspondingly,

$$u = \bar{\mathcal{D}}\bar{p}$$

where u is the m^n by 1 vector of $u_{i,j,\ldots,k}$, $\bar{p} = \mathcal{C}p$ is a reduced parameter vector, p is the $m*n$ by 1 vector of $\alpha_i, \beta_j, \ldots, \xi_k$, and

$$\mathcal{C} = \begin{bmatrix} I_m & J_{m,m} & J_{m,m} & \cdots & \cdots & J_{m,m} \\ O_{m-1,m} & G_{m-1,m} & O_{m-1,m} & \cdots & \cdots & O_{m-1,m} \\ O_{m-1,m} & O_{m-1,m} & G_{m-1,m} & \cdots & \cdots & O_{m-1,m} \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ O_{m-1,m} & O_{m-1,m} & O_{m-1,m} & \cdots & \cdots & G_{m-1,m} \end{bmatrix}$$

Here, I_i is the i by i identity matrix, $O_{i,j}$ is the i by j null matrix, $J_{m,m} = [O_{m,m-1} \quad 1(m,1)]$, $G_{m-1,m} = [I_{m-1} \quad -1(m,1)]$ and $1(i,1)$ is the i by 1 column vector of 1's. Similarly, a solution for $\bar{p}$ is given by the corresponding equation of (3.13). There are now $n-1$ free parameters $\beta_m, \ldots, \xi_m$ in p. For convenience, they can all be set to zero in the determination of $\alpha_i, \beta_j, \ldots, \xi_k$ from $\bar{p}$.

The database of the subsystem now consists of the parameter $\bar{p}$ instead of the hyper-cube as in the case of original fuzzy system. The number of data stored is $m * n$ instead of m^n. The database is substantially reduced.

From the above discussion, decomposition of the fuzzy system depends only on the values of the consequence $u_{i,j}$s. It can be observed from eqn. (3.4) to eqn. (3.7) that the antecedent membership functions do not play a role. They come into play only at subsystem inference level.

3.5 Decomposable Approximation

In case that $u_{i,j}$s in the original system (3.1) do not satisfy "Lattice" property, the above process can still be adopted for finding a decomposable approximation with consequents $u^*_{i,j}$s, which satisfy the "Lattice" property, i.e. there would exist corresponding α^*_i, β^*_j such that

$$u^*_{i,j} = \alpha^*_i + \beta^*_j$$

The approximation $u^*_{i,j}$s should be close to $u_{i,j}$ in some sense. One popular criterion would be that $u^*_{i,j}$s minimize the square error cost function [36, 37]

$$J = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}(u_{i,j} - u^*_{i,j})^2$$

Specifically, one can pose a least square error problem as follows: *find parameter vector* $p^*(= [\alpha^*_i, \beta^*_j]^T, i, j = 1, \ldots, 3)$ *such that* $J = 1/2e^T e$ *is minimized, where* $e = (u - u^*)$ *and* $u^* = \mathcal{D}p^*$.

In the general case of n fuzzy variables and each with m linguistic terms. The cost function becomes,

$$J = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\cdots\sum_{k=1}^{m}(u_{i,j,\ldots,k} - u^*_{i,j,\ldots,k})^2 \tag{3.14}$$

Solution to the above least square problem is standard [36], and turns out to be the same as using $\bar{p}^* = \bar{\mathcal{D}}^{\dagger}u$ to determine the reduced parameter vector $\bar{p}^*$ and then obtain p^* from $\bar{p}^*$ through assignment of free parameter. The original non-decomposable fuzzy system can hence be approximated as the sum of subsystems

$$\begin{array}{lll}
\text{Subsystem} \quad X_1 : & \text{If } X_{1,i} \Rightarrow \delta(\alpha_i^*), & \text{for } i = 1, \ldots, m \\
\text{Subsystem} \quad X_2 : & \text{If } X_{2,j} \Rightarrow \delta(\beta_j^*), & \text{for } j = 1, \ldots, m \\
\vdots & \vdots & \\
\text{Subsystem} \quad X_n : & \text{If } X_{n,k} \Rightarrow \delta(\xi_k^*), & \text{for } k = 1, \ldots, m
\end{array} \tag{3.15}$$

It is sometimes desirable that the approximation (3.15) satisfy some additional conditions. The problem becomes the so called constrained least-squares estimation problem [36] which is to minimizes the cost function (3.14) subjected to some constraint on p^*,

$$Fp^* - G = 0 \tag{3.16}$$

Solution for this constrained problem, denoted as p_c^*, can be obtained by first computing the non-constrained decomposable approximation p^* as above and then incorporating the constraints by

$$p_c^* = p^* - \bar{\mathcal{D}}^\dagger F^T (F \bar{\mathcal{D}}^\dagger F^T)^{-1} (Fp^* - G) \tag{3.17}$$

For example, with $n = 2$, $m = 3$, if the condition $u_{2,3} = 2$ is to be imposed, F and G in the constraint (3.16) are given by $F = [010001]$, and $G = 2$.

3.5.1 Linear Approximation

If the membership functions for each fuzzy inputs are triangular in shape and at most two membership functions overlapping with a sum total of membership degree of 1, the original system may be approximated by a linear system.

Let's consider decomposed subsystem of X_1. For any input $x_{1,o} \in [x_{1,i}, x_{1,i+1}]$, rule i and $i+1$ are fired with $\mu_{X_{1,i}}(x_{1,o}) = 1 - \frac{x_{1,o}-x_{1,i}}{x_{1,i+1}-x_{1,i}}$, $\mu_{X_{1,i+1}}(x_{1,o}) = 1 - \mu_{X_{1,i}}(x_{1,o}) = \frac{x_{1,o}-x_{1,i}}{x_{1,i+1}-x_{1,i}}$. Here, $x_{1,i}$ and $x_{1,i+1}$ are, respectively, locations of the ith and $i+1$th membership function for x. The output of the subsystem is

$$\begin{aligned} u_{X,1}(x_{1,o}) &= \alpha_i^* * (1 - \frac{x_{1,o} - x_{1,i}}{x_{1,i+1} - x_{1,i}}) + \alpha_{i+1}^* * \frac{x_{1,o} - x_{1,i}}{x_{1,i+1} - x_{1,i}} \\ &= \alpha_i^* + (\alpha_i^* - \alpha_{i+1}^*) * \frac{x_{1,o} - x_{1,i}}{x_{1,i+1} - x_{1,i}} \end{aligned} \tag{3.18}$$

It can be seen that eqn.(3.18) defines a line segment between $x_{1,i}$ and $x_{1,i+1}$ joining α_i^* at $x_1 = x_{1,i}$ and α_{i+1}^* at $x_1 = x_{1,i+1}$. The slope of this line is $\frac{\alpha_i^*-\alpha_{i+1}^*}{x_{1,i+1}-x_{1,i}}$. The subsystem defines a straight line if all the line segments have same slope, i.e.

$$\frac{\alpha_i^* - \alpha_{i+1}^*}{x_{1,i+1} - x_{1,i}} = \frac{\alpha_{i+1}^* - \alpha_{i+2}^*}{x_{1,i+2} - x_{1,i+1}} \tag{3.19}$$

for $i = 1, 2, \ldots, m_1 - 2$. Similar expressions hold for subsystems $X_2, \ldots, X_n$.

The above constrained approximation results in an affine (linear plus a constant) system. Linear approximation could be given if the constraint that zero input for zero output is further imposed. From eqn. (3.18), output of subsystem X_1 at $x_{1,o} = 0$ is

$$u_{X,1}(0) = \alpha_i^* - \frac{(\alpha_i^* - \alpha_{i+1}^*)}{x_{1,i+1} - x_{1,i}} x_{1,i}$$

Confining the additive output at $x_{1,o} = 0$, $x_{2,o} = 0, \ldots$, $x_{n,o} = 0$ to zero yields

$$\begin{aligned}(\alpha_i^* - \frac{(\alpha_i^* - \alpha_{i+1}^*)}{x_{1,i+1} - x_{1,i}} x_{1,i}) + (\beta_j^* - \frac{(\beta_j^* - \beta_{j+1}^*)}{x_{2,j+1} - x_{2,j}} x_{2,j}) \\ + \cdots + (\xi_k^* - \frac{(\xi_k^* - \xi_{k+1}^*)}{x_{n,k+1} - x_{n,k}} x_{n,k}) = 0\end{aligned} \tag{3.20}$$

Eqn. (3.19) and (3.20) together are the constraints for linear control. Note that in eqn. (3.20), $i, j, \ldots, k$ can be take on any integer values between $1, \ldots, m-1$ with the same effects.

3.5.2 Case Study

Consider the system with two fuzzy variables each containing three linguistic terms. The fuzzy system has triangular membership functions X_i and Y_j located at $x_1 = -1.2$, $x_2 = 0$, $x_3 = 1.2$ and $y_1 = -1.6$, $y_2 = 0$, $y_3 = 1.6$. The x_i and y_j are evenly spaced for this example but they do not have to be. The fuzzy rules are as given in eqn. (3.1). Table 3.1 tabulates $u_{i,j}$s. It can be observed that matrix $\mathcal{U}$ does not satisfy the "lattice" property and the system is not decomposable.

	X_1	X_2	X_3
Y_1	-11.0	-5.0	2.0
Y_2	-5.0	0	6.0
Y_3	-1.0	5.0	10.0

Table 3.1: Locations $u_{i,j}$ of Original Fuzzy System

	X_1	X_2	X_3
Y_1	-10.44	-4.78	1.22
Y_2	-5.44	0.22	6.22
Y_3	-1.11	4.56	10.56

Table 3.2: Locations $u^*_{i,j}$ of Decomposable Approximation

Least square minimization to determine a decomposable approximation is conducted. Using $\bar{p}^* = \bar{\mathcal{D}}^\dagger u$, the reduced parameter vector is determined as $\bar{p}^*$

= [-1.11 4.56 10.56 -9.33 -4.33]^T. Hence, $\alpha_1^* = -1.11$, $\alpha_2^* = 4.56$, $\alpha_3^* = 10.56$, $\beta_1^* = -9.33$, $\beta_2^* = -4.33$, and $\beta_3^* = 0$ (free parameter). The approximated system is an additive fuzzy subsystems of X and Y as given by

Subsystem X	Rule i: If $X_i \Rightarrow \delta(\alpha_i^*)$,	for $i = 1, 2, 3$
Subsystem Y	Rule j: If $Y_j \Rightarrow \delta(\beta_j^*)$,	for $j = 1, 2, 3$

The values α_i^* and β_j^* indicate that the inference of subsystem corresponds to two line segments with different slopes and the decomposable approximation is nonlinear. The values of $u_{i,j}^* = \alpha_i^* + \beta_j^*$ are tabulated in Table 3.2. Fig. 3.1 compares the output profiles of the original fuzzy system and its decomposable approximation. The decomposed system does not have the same stable point (0,0) as the original system. If the stable point (0,0) need to be reserved, the corresponding constrain that $\alpha_2^* + \beta_2^* = 0$ should be imposed on the least square minimization process.

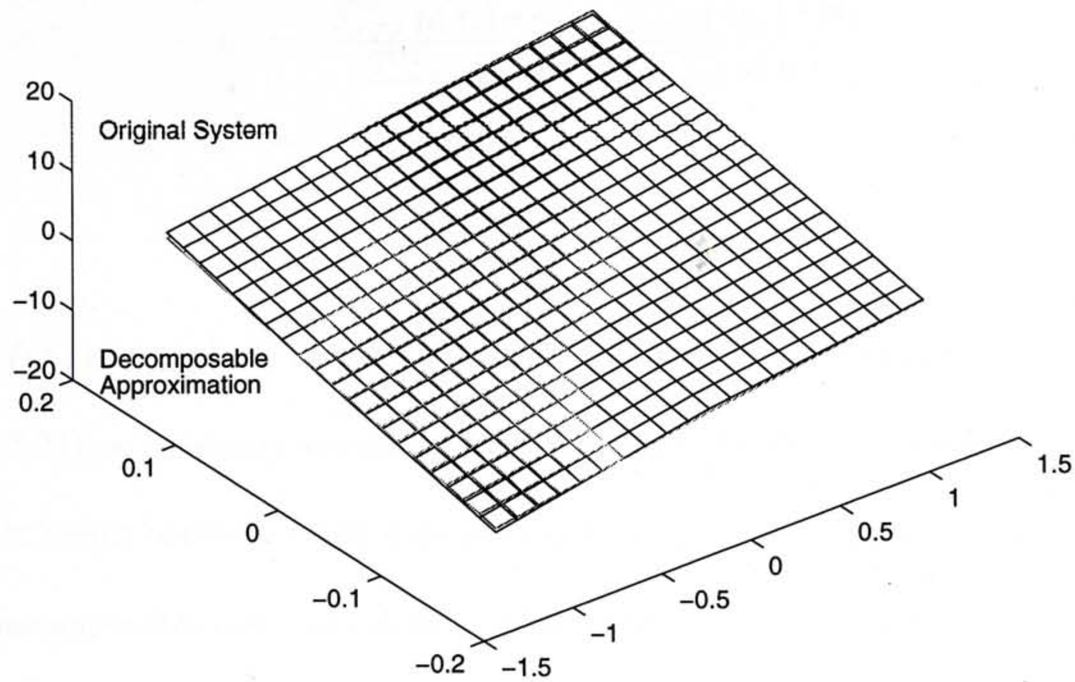


Fig 3.1. Output Profiles: Original System Vs Decomposable Approximation

3.6 Limitation of Decomposable Approximation

Section 3.5 conducts LSE approximation of a fuzzy system by additive fuzzy subsystems. This section focuses on the quality of the decomposable approximation.

For illustration, let's consider a fuzzy system with two input variables x and y each of which has m linguistic terms. Let the system be decomposable, i.e., the overall output of the system u for an input (x, y) can be expressed as:

$$
\begin{aligned}
u &= \frac{\sum_{i=1}^{m} \mu(x_i) * \alpha_i}{\sum_{i=1}^{m} \mu(x_i)} + \frac{\sum_{j=1}^{m} \mu(y_j) * \beta_j}{\sum_{j=1}^{m} \mu(y_j)} \\
&= f(x) + g(y) \qquad (3.21)
\end{aligned}
$$

Here, $f(x)$ and $g(y)$ are functions of x and y as given in the equation. In view of eqn. (3.21), a necessary condition for a system to be decomposable is that the product terms between x and y do not exist. Hence, if a system is approximated by a decomposable one, the decomposable approximation would have large error if the original system have large cross product terms.

The above discussion can be generalized to system with n fuzzy variables $(x_1, x_2, \ldots, x_n)$. The overall output for a decomposable system is now

$$
\begin{aligned}
u &= \frac{\sum_{i=1}^{m} \mu(x_{1,i}) * \alpha_i}{\sum_{i=1}^{m} \mu(x_{1,i})} + \frac{\sum_{j=1}^{m} \mu(x_{2,j}) * \beta_j}{\sum_{j=1}^{m} \mu(x_{2,j})} + \cdots + \frac{\sum_{k=1}^{m} \mu(x_{n,k}) * \xi_k}{\sum_{k=1}^{m} \mu(x_{n,k})} \\
&= f(x_1) + g(x_2) + \cdots + h(x_n) \qquad (3.22)
\end{aligned}
$$

Again, there are no cross-product terms in eqn. (3.22). Hence, decomposable approximation would be satisfactory if the original system has weak linkage among different variables.

3.7 Approximation Index

As mathematical expression of the original fuzzy system are in general not available, one could not tell whether the variables of the original system are weakly linked or not. In this section, an approximation index is proposed as an indicator to how close the decomposable approximation is to the original system.

A common indicator of approximation that comes to mind is the mean-squared-error between rule consequents $u_{i,j}$ and $u^*_{i,j}$ (see eqn. (3.23)).

$$E_{ms} = \sum_{i=1}^{m}\sum_{j=1}^{m}(u_{i,j} - u^*_{i,j})^2 \tag{3.23}$$

where, m is the number of linguistic terms defined for each fuzzy input variable, $u_{i,j}$ and $u^*_{i,j}$ are the fuzzy rule consequents for the original system and the decomposable approximation respectively. It turns out that, however, E_{ms} is not a suitable indicator. Consider two systems each having two fuzzy input variables. Let the regions of interest be [-1, 1] for both input variables in the first system and, correspondingly, [-2, 2] in the second system. Consequents $u_{i,j}$ of the systems are shown in Table 3.3 and $u^*_{i,j}$ of their decomposable approximation in Table 3.4.

The consequents of system 2 differ from that of system 1 by a scaling factor

-12	-8	-5	-2	2
-9	-6	-2	1	5
-4	-3	0	4	7
-2	0	2	7	10
1	4	7	11	13

(a) System 1

-24	-16	-10	-4	4
-18	-12	-4	2	10
-8	-6	0	8	14
-4	0	4	14	20
2	8	14	22	26

(b) System 2

Table 3.3: The Locations of $u_{i,j}$ for the Two Given Systems

-11	-8.4	-5.6	-1.6	1.6
-8.2	-5.6	-2.8	1.2	4.4
-5.4	-2.8	0	4	7.2
-2.6	0	2.8	6.8	10
1.2	3.8	6.6	10.6	13.8

(a) Approximation of System 1

-22	-16.8	-11.2	-3.2	3.2
-16.4	-11.2	-5.6	2.4	8.8
-10.8	-5.6	0	8	14.4
-5.2	0	5.6	13.6	20
2.4	7.6	13.2	21.2	27.6

(b) Approximation of System 2

Table 3.4: The Locations of $u^*_{i,j}$ for the Decomposable Approximation

of 2. The mean-squared-error (E_{ms}) for system 1 and system 2 are 0.312 and 1.248 respectively. The mean-squared-error is a factor of 4 larger in system 2 but in reality, the approximation is of the same quality for both. This hence indicates that mean-squared-error is not a "good" approximation index.

An alternative is the mean-square-percentage-error as defined by:

$$E_{msp} = \frac{1}{m * m} \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\frac{u_{i,j} - u^*_{i,j}}{u_{i,j}}\right)^2 \qquad u_{i,j} \neq 0 \tag{3.24}$$

For the example considered in Tables 3.3 and 3.4, the mean-squared-percentage-error equals 0.033 for both systems. Note that eqn. (3.24) sums only the terms with nonzero $u_{i,j}$. In case the original system contains rules with zero consequents, eqn. (3.24) may not give accurate indication as to the quality of approximation. To solve this problem, the mean-squared-percentage-error, E_{mspc}, of the differences in value between the adjacent cells is proposed in the present work

$$E_x = \frac{1}{m*(m-1)} \sum_{i=1}^{m-1} \sum_{j=1}^{m} \left(\frac{(u_{i+1,j} - u_{i,j}) - (u^*_{i+1,j} - u^*_{i,j})}{(u^*_{i+1,j} - u^*_{i,j})}\right)^2 \qquad u^*_{i+1,j} \neq u^*_{i,j} \tag{3.25}$$

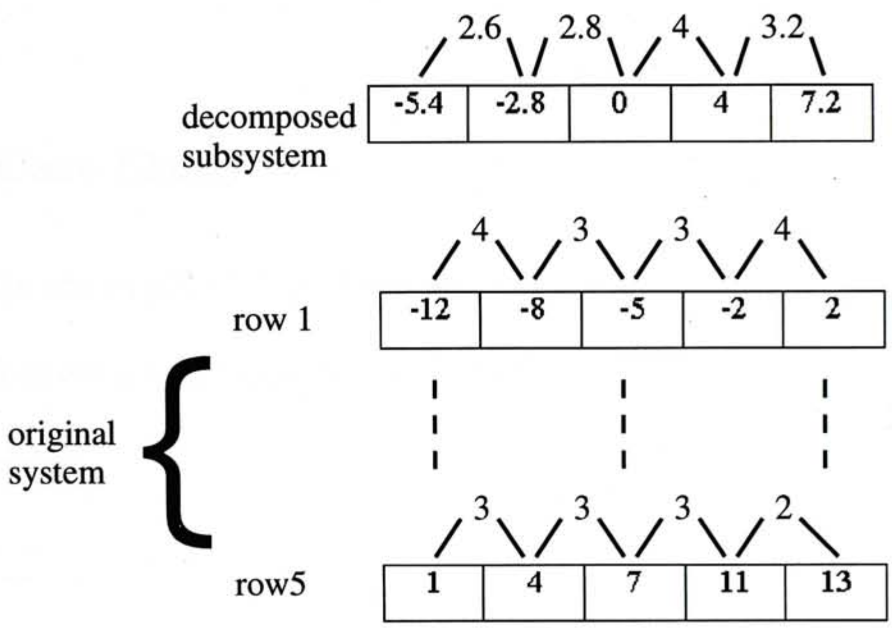
(3.26)

$$E_y = \frac{1}{(m-1)*m} \sum_{i=1}^{m} \sum_{j=1}^{m-1} \left(\frac{(u_{i,j+1} - u_{i,j}) - (u^*_{i,j+1} - u^*_{i,j})}{(u^*_{i,j+1} - u^*_{i,j})}\right)^2 \qquad u^*_{i,j+1} \neq u^*_{i,j} \tag{3.27}$$

and

$$E_{mspc} = \frac{1}{2}(E_x + E_y) \tag{3.28}$$

The denominators of eqn. (3.25) and (3.27) involve the consequents from the decomposable approximation. As decomposable approximation satisfies "Lattice" property, adjacent rows or columns of $u^*_{i,j}$ are not as likely to be equal and yield zero values in the denominators of eqn. (3.25) and eqn. (3.27). For the example considered, $E_x = 0.0633$, $E_y = 0.0755$ and $E_{mspc} = 0.0694$ for both systems. The process of determinating the error (E_x) is illustrated in Fig. 3.2. Note that the difference [2.6, 2.8, 4, 3.2] can be obtained using any row of the decomposable system (see Table 3.4(a)).



$$E_x = \left(\frac{2.6-4}{2.6}\right)^2 + \left(\frac{2.8-3}{2.8}\right)^2 + \ldots\ldots + \left(\frac{3.2-2}{3.2}\right)^2$$
$$= 0.0633$$

Fig 3.2 The Process of determinating the Error

Generalizing, the E_{mspc} for a fuzzy system with n inputs $(x_1, x_2, \ldots, x_n)$ is

$$E_{mspc} = \frac{\sum_{j=1}^{n} E_{x_j}}{n} \tag{3.29}$$

where E_{x_j} is the error with respect to the fuzzy variable x_j and is given as:

$$E_{x_j} = \frac{1}{m^{n-1} * (m-1)} \sum_{i=1}^{m} \cdots \sum_{j=1}^{m-1} \cdots \sum_{k=1}^{m} \left(\frac{(u_{i+1,j} - u_{i,j}) - (u^*_{i+1,j} - u^*_{i,j})}{(u^*_{i+1,j} - u^*_{i,j})} \right)^2$$

$$u^*_{i,..,j+1,..,k} \neq u^*_{i,..,j,..,k}$$

3.7.1 Case Study

To illustrate the applicability of the proposed approximation index E_{mspc}, consider the following non-linear functions with various degree of coupling among input variables:

$$g_1 = 2x^2 + 3sin^2(y) \tag{3.30}$$

$$g_2 = (2x^2 + 3sin^2(y))^2 \tag{3.31}$$

$$g_3 = 4x^2 + 2y + xsin(xy) \tag{3.32}$$

$$g_4 = x + 3x^2 + 4xy + 2xy^2 + y^2 \tag{3.33}$$

Consider four fuzzy systems of x and y each containing 5 linguistic terms with triangular membership functions located at (-1, -0.5, 0, 0.5, 1) for both x and y. Let the rule consequents for the systems be generated by functions g_1, g_2, g_3

and g_4 at the grid points of the membership function locations and are as shown in Table 3.5 (a), (b), (c), and (d), respectively.

4.12	2.69	2.00	2.69	4.12
2.62	1.19	0.50	1.19	2.62
2.12	0.69	0	0.69	2.12
2.62	1.19	0.50	1.19	2.62
4.12	2.69	2.00	2.69	4.12

(a)

17.01	7.23	4.00	7.24	17.01
6.89	1.42	0.25	1.42	6.89
4.51	0.48	0	0.48	4.51
6.89	1.42	0.25	1.42	6.89
17.01	7.23	4.00	7.23	17.01

(b)

1.16	2.52	4.00	5.48	6.84
-1.24	-0.12	1.00	2.12	3.24
-2.00	-1.00	0	1.00	2.00
-1.24	-0.12	1.00	2.12	3.24
1.16	2.52	4.00	5.48	6.84

(c)

5.00	3.75	2.00	-0.25	-3.00
2.50	1.25	0.25	-0.75	-1.75
1.00	0.25	0	0.25	1.00
1.25	0.75	1.25	2.75	5.25
3.00	2.75	4.00	6.75	11.00

(d)

Table 3.5: . Rule Consequent $u_{i,j}$ for the Fuzzy Systems

The decomposed fuzzy subsystems and corresponding value for E_{mspc} are shown in Table 3.6 (a), (b), (c), and (d), respectively. Profiles of the fuzzy subsystems using triangular membership functions as compared to the original functions are shown in Fig 3.3.

It is observed from Fig. 3.3 that the decomposed fuzzy inference approximation of the fuzzy systems as generated by g_1 and g_3 are good approximation to the original systems as they have zero or small cross terms. The same, however, cannot be said for the cases of fuzzy systems generated by g_2 and g_4 as they involve large cross product terms. The observations from Fig. 3.3 is consistent with the values of the approximation index E_{mspc} as shown in Table 3.6.

2.00
0.50
0
0.50
2.00

$E_y = 0$

2.12	0.69	0	0.69	2.12

$E_X = 0$
$E_{mspc} = 0$
(a)

7.25
0.12
0
0.12
7.25

$E_y = 4.0284$

7.51	0.60	0	0.60	7.51

$E_X = 73.4829$
$E_{mspc} = 38.7557$
(b)

4.00
1.00
0
1.00
4.00

$E_y = 0.0204$

-2.43	-1.24	0	1.24	2.43

$E_X = 0.0254$
$E_{mspc} = 0.0229$
(c)

1.00
-0.25
0
1.75
5.00

$E_y = 19.7778$

1.00	0.25	0	0.25	1.00

$E_X = 9.2770$
$E_{mspc} = 14.5274$
(d)

Table 3.6: The Decomposed Subsystems and Errors
for the Given Fuzzy Systems

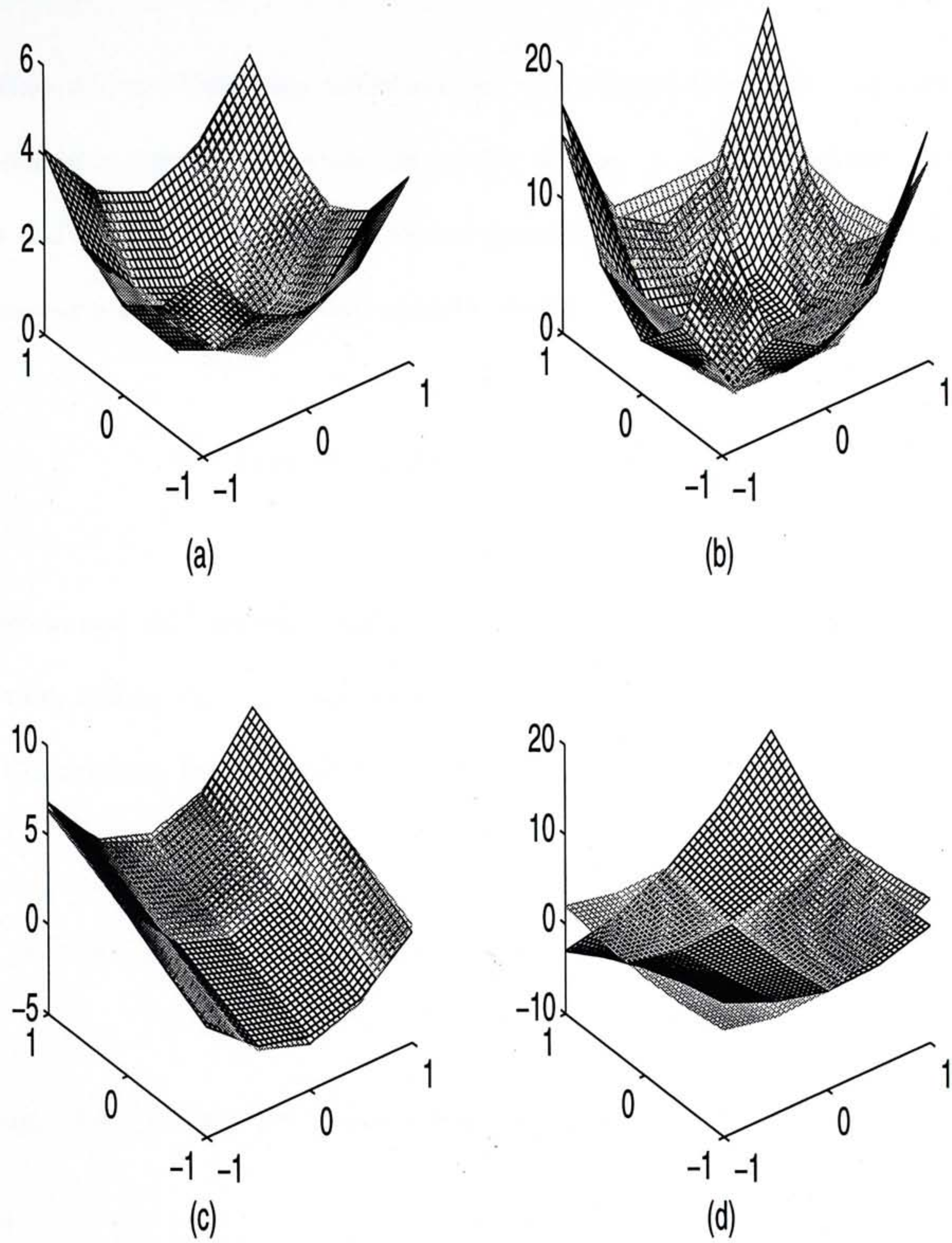


Fig 3.3 Output Profiles of the Given Systems Vs Their Decomposed Approximation

3.8 Decomposable TSK Model

Sections 3.2 to 3.5 describe decomposition of the linguistic model. The concept developed in these sections can be readily applied to the TSK model. Recall that the output of the TSK model is a general function of the input variables instead of a single value in the linguistic model,

$$u_i = f_i(x_1, x_2, \ldots, x_n) \qquad \text{for} \quad i = 1, 2, \ldots, m \tag{3.34}$$

where u_i and $f_i(.)$ are the respective output value and output function for the ith rule, and $x_1, x_2, \ldots, x_n$ are the input values.

Consider the linear case of the TSK model. Eqn. (3.34) can be expressed as

$$u_i = a_{i,o} + a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n \qquad \text{for} \quad i = 1, 2, \ldots, m \tag{3.35}$$

If eqn. (3.35) is rewritten in vector form, we have

$$u_i = [a_{i,o} \quad a_{i,1} \quad a_{i,2} \ \ldots \ a_{i,n}] * [1 \quad x_1 \quad x_2 \ \ldots \ x_n]^T \tag{3.36}$$

It is seen that the coefficient vector, $[a_{i,o}\ a_{i,1}\ a_{i,2} \ldots\ a_{i,n}]$, for $i = 1, 2, \ldots, m$, for each rule is multiplied by the same variable vector $[1\ x_1\ x_2\ \ldots\ x_n]^T$. The TSK model could be modified as

$$
\begin{array}{llll}
\text{IF } x_1 \text{ is } X_{1,1}, & \ldots & , x_n \text{ is } X_{1,n} \Rightarrow & u'_1 = [a_{1,o}\, a_{1,1}\, \ldots\, a_{1,n}] \\
\vdots & \vdots & \vdots & \vdots \\
\text{IF } x_1 \text{ is } X_{m,1}, & \ldots & , x_n \text{ is } X_{m,n} \Rightarrow & u'_m = [a_{m,o}\, a_{m,1}\, \ldots\, a_{m,n}]
\end{array}
\tag{3.37}
$$

with overall output

$$
u = (\sum_{i=1}^{m} \mu_i * u'_i) * [1 \quad x_1 \quad x_2 \ \ldots \ x_n]^T \tag{3.38}
$$

The modified TSK model in eqn. (3.37) can be viewed as a linguistic fuzzy model with vector linguistic output. The decomposition process can be conducted by considering each element of the output individually. The approach is also applicable to nonlinear case. For example, if eqn. (3.35) has an additional cross term of $a_i^c x_1 x_2$, then eqn. (3.36) would be modified as

$$
u_i = [a_{i,o} \quad a_{i,1} \quad a_{i,2} \ \ldots \ a_{i,n} \quad a_i^c] * [1 \quad x_1 \quad x_2 \ \ldots \ x_n \quad x_1 x_2]^T
$$

3.8.1 Case Study

Consider a fuzzy TSK model with two fuzzy variables each fuzzy variable containing three linguistic and the fuzzy rules are given as follows:

$$
\begin{array}{lll}
\text{Rule 1:} & \text{IF } x \text{ is } X_1,\ y \text{ is } Y_1, \Rightarrow & u_1 = -10 + 2.3xy - 3.2y \\
\text{Rule 2:} & \text{IF } x \text{ is } X_1,\ y \text{ is } Y_2, \Rightarrow & u_2 = -3 + 5.1xy - y \\
\text{Rule 3:} & \text{IF } x \text{ is } X_1,\ y \text{ is } Y_3, \Rightarrow & u_3 = 6 + 7.8xy + y \\
\text{Rule 4:} & \text{IF } x \text{ is } X_2,\ y \text{ is } Y_1, \Rightarrow & u_4 = -5 + 4.4xy + 0.5y \\
\text{Rule 5:} & \text{IF } x \text{ is } X_2,\ y \text{ is } Y_2, \Rightarrow & u_5 = 1 + 6.9xy + 2.5y \\
\text{Rule 6:} & \text{IF } x \text{ is } X_2,\ y \text{ is } Y_3, \Rightarrow & u_6 = 8 + 8.2xy + 4.5y \\
\text{Rule 7:} & \text{IF } x \text{ is } X_3,\ y \text{ is } Y_1, \Rightarrow & u_7 = -2 + 5.5xy + 2y \\
\text{Rule 8:} & \text{IF } x \text{ is } X_3,\ y \text{ is } Y_2, \Rightarrow & u_8 = 4 + 9.2xy + 3.8y \\
\text{Rule 9:} & \text{IF } x \text{ is } X_3,\ y \text{ is } Y_3, \Rightarrow & u_9 = 12 + 11.4xy + 6.5y
\end{array}
\tag{3.39}
$$

The TSK model (3.39) can be converted into vector output format as shown in eqn. (3.37).

$$
\begin{array}{lll}
\text{Rule 1:} & \text{IF } x \text{ is } X_1,\ y \text{ is } Y_1, \Rightarrow & u'_1 = [-10 \quad 2.3 \quad 3.2] \\
\text{Rule 2:} & \text{IF } x \text{ is } X_1,\ y \text{ is } Y_2, \Rightarrow & u'_2 = [-3 \quad 5.1 \quad -1] \\
\text{Rule 3:} & \text{IF } x \text{ is } X_1,\ y \text{ is } Y_3, \Rightarrow & u'_3 = [6 \quad 7.8 \quad 1] \\
\text{Rule 4:} & \text{IF } x \text{ is } X_2,\ y \text{ is } Y_1, \Rightarrow & u'_4 = [-5 \quad 4.4 \quad 0.5] \\
\text{Rule 5:} & \text{IF } x \text{ is } X_2,\ y \text{ is } Y_2, \Rightarrow & u'_5 = [1 \quad 6.9 \quad 2.5] \\
\text{Rule 6:} & \text{IF } x \text{ is } X_2,\ y \text{ is } Y_3, \Rightarrow & u'_6 = [8 \quad 8.2 \quad 4.5] \\
\text{Rule 7:} & \text{IF } x \text{ is } X_3,\ y \text{ is } Y_1, \Rightarrow & u'_7 = [-2 \quad 5.5 \quad 2] \\
\text{Rule 8:} & \text{IF } x \text{ is } X_3,\ y \text{ is } Y_2, \Rightarrow & u'_8 = [4 \quad 9.2 \quad 3.8]
\end{array}
\tag{3.40}
$$

$$\text{Rule 9:} \qquad \text{IF } x \text{ is } X_3,\ y \text{ is } Y_3,\ \Rightarrow\ u'_9 = [12 \quad 11.4 \quad 6.5]$$

Decomposition process can hence be carried out term by term. The fuzzy rules corresponding to the various terms are tabulated in Table 3.7 and their decomposed counterparts are tabulated in Table 3.8.

	X_1	X_2	X_3
Y_1	-10	-5	2
Y_2	-3	1	4
Y_3	-6	8	12

(a) constant term

	X_1	X_2	X_3
Y_1	2.3	4.4	5.5
Y_2	5.1	6.9	9.2
Y_3	7.8	8.2	11.4

(b) x term

	X_1	X_2	X_3
Y_1	-3.2	0.5	2
Y_2	-1	2.5	3.8
Y_3	1	4.5	6.5

(c) y term

Table 3.7: Rule Consequents of the TSK model

	α_i	β_i
$i=1$	5.11	-14.33
$i=2$	8.78	-8.00
$i=3$	12.11	0

(a) constant term

	α_i	β_i
$i=1$	7.44	-5.07
$i=2$	8.88	-2.07
$i=3$	11.08	0

(b) x term

	α_i	β_i
$i=1$	1.09	-4.23
$i=2$	4.66	-2.23
$i=3$	6.26	0

(c) y term

Table 3.8: Parameter α_i and β_j of the decomposed TSK model

Fuzzy system (3.39) is hence decomposed into 2 additive fuzzy subsystems X and Y given as follows:

$SubsystemX$

Rule 1:	IF x is X_1, $\Rightarrow$ $u_{X1} = 5.11 + 7.44xy + 1.09y$
Rule 2:	IF x is X_2, $\Rightarrow$ $u_{X2} = 8.78 + 8.88xy + 4.66y$
Rule 3:	IF x is X_3, $\Rightarrow$ $u_{X3} = 12.11 + 11.08xy + 6.26y$

$SubsystemY$

Rule 1:	IF y is Y_1, $\Rightarrow$ $u_{Y1} = -14.33 - 5.07xy - 4.23y$
Rule 2:	IF y is Y_2, $\Rightarrow$ $u_{Y2} = -8.00 - 2.07xy - 4.23y$
Rule 3:	IF y is Y_3, $\Rightarrow$ $u_{Y3} = 0$

3.9 Conclusion

In this chapter, conditions for decomposing a fuzzy system into additive subsystems are studied. The decomposed system has the advantage of requiring a reduced database than the original one. The decomposition conditions depend on the values of the consequence of the fuzzy rules. The antecedent membership functions come into play only at subsystem inference.

For a general fuzzy system, an approach to obtain decomposable approximation based on square error minimization is proposed. Limitation of the decomposable approximation is also discussed. A good approximation will be achieved when the original fuzzy inference possesses small cross product terms. As the inference function of a fuzzy system may not be readily known, an approximation index is introduced. A simulation example is conducted to illustrate computation of the proposed approximation index.

Chapter 4

Fuzzy Identification

4.1 Introduction

Fuzzy system is generally a nonlinear system. The overall characteristic of the system is affected by locations of the rules, the inference parameter and the membership functions. By choosing these three components appropriately, a fuzzy system is capable of approximating any function [38, 39, 40].

System identification based upon input-output data is a classical approximation problem in system theory. The conventional identification strategy usually assumes that the global characteristic of a system can be represented by a simple function [38, 39, 40]. However, such a global assumption is difficult to satisfy especially for highly nonlinear systems. Some studies [29, 32, 33, 34] have shown that fuzzy system seems to be more appropriate model for some nonlinear systems.

This chapter aims to study the feasibility and problems encountered in system identification of the following kinds of fuzzy models: linguistic model, linear TSK model, and decomposable subsystems model. Conventional least-squares estimation technique is utilized. The relationship between fuzzy system identification and weighted least-squares errors identification is also discussed, as well as the concept of sub-dividing the whole system into regional subsystems for identification. Some examples to illustrate details of the present approach are given. Throughout the chapter, all systems considered are Product-Sum-Gravity inference based, with at most two antecedent membership overlapping each other and sum of membership degrees equals one.

4.2 Least-squares Estimation

The least-squares estimation aims to minimize the sum of the squared errors between outputs of the identified model and the experimental data. Consider the linear model

$$u = \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_n x_n \tag{4.1}$$

where $(x_1, x_2, \ldots, x_n)$ is a set of n linearly independent variables, and $(\theta_1, \theta_2, \ldots, \theta_n)$ is a set of constant parameters to be determined. In order to determine the n parameters, h input-output observations, with $h \geq n$, are required. The observation can be expressed as a set of h linear equations:

$$u(i) = \theta_1 x_1(i) + \theta_2 x_2(i) + \ldots + \theta_n x_n(i) + e(i) \tag{4.2}$$

where, $i = 1, 2, \ldots, h$, and $e(i)$ is the error in the ith observation. Equation (4.2) can be represented in matrix form

$$U = \Phi\Theta + \epsilon$$

or,

$$\epsilon = U - \Phi\Theta \tag{4.3}$$

where

$$U = \begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(h) \end{bmatrix}, \Phi = \begin{bmatrix} x_1(1) & x_2(1) & \cdots & x_n(1) \\ x_1(2) & x_2(2) & \cdots & x_n(2) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(h) & x_2(h) & \cdots & x_n(h) \end{bmatrix}, \Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}, \epsilon = \begin{bmatrix} e(1) \\ e(2) \\ \vdots \\ e(h) \end{bmatrix}.$$

For least-squares estimation (LSE), the estimate of Θ, $\hat{\Theta}$, is chosen such that the cost function

$$J = \epsilon^T \epsilon = (U - \Phi\Theta)^T (U - \Phi\Theta) \tag{4.4}$$

is minimized. The optimal estimation $\hat{\Theta}$ is obtained as [36]

$$\hat{\Theta} = (\Phi^T \Phi)^{-1} \Phi^T U. \tag{4.5}$$

The above result is derived from a cost function J which weights every error component equally. The LSE formulation can be generalized to allow each error term to be weighted differently, i.e., $J = \epsilon^T W \epsilon$, and the solution is [36],

$$\hat{\Theta}_w = (\Phi^T W \Phi)^{-1} \Phi^T W U. \tag{4.6}$$

where $W = diag(w(i))$, $i = 1, \ldots, h$, is a h by h diagonal weighting matrix.

It is observed from eqn. (4.5) and eqn. (4.6) that the estimation process involves the inverse matrix, $(\Phi^T\Phi)^{-1}$ and $(\Phi^T W\Phi)^{-1}$. The inverse matrix exists if the matrix, Φ, is full rank. That means the column vector constituting Φ must be linearly independent; otherwise, the estimation problem may need to reformulated.

Example 4.1: Consider, for example, a system with two input: $u(i) = \theta_1 x(i) + \theta_2 y(i)$. Let **12** sets of input be randomly drawn between [-1,1] for both $x(i)$ and $y(i)$, $i = 1, 2, \ldots, 12$, resulting in

$$\Phi = \begin{bmatrix} 0.05 & -0.81 & 0.30 & -0.17 & 0.44 & 0.82 & -0.91 & 0.67 & -0.34 & 0.26 & 0.52 & -0.24 \\ -0.85 & 0.26 & 0.77 & -0.45 & -0.13 & 0.53 & -0.39 & -0.28 & 0.14 & -0.03 & 0.88 & 0.62 \end{bmatrix}^T$$

It is seen that the rank of $\Phi = 2$. The matrix, $\Phi^T\Phi$, is invertible and the inverse matrix

$$(\Phi^T\Phi)^{-1} = \begin{bmatrix} 0.31 & -0.08 \\ -0.08 & 0.33 \end{bmatrix}$$

LSE can hence be carried out for this system.

4.3 LSE Formulation of Various Fuzzy Models

In traditional system identification problem, input variables can be excited randomly as in Example 4.1 and hence are likely to be linearly independent. In fuzzy system, however, the input variables need to be converted first into membership degrees according to the locations and profiles of the membership functions. As will be shown later, this may result in that the number of parameters to be estimated are much larger than that of the original input variables. One should hence be cautious to check if there exists linear dependency in matrix Φ. If so, this dependency must be eliminated before the identification is conducted.

4.3.1 Linguistic Model

For illustration, consider a linguistic fuzzy system of two inputs each having m linguistic terms. The membership functions are assumed normal with at most two overlapping each other and sum of membership degrees equals one. The general rule is:

$$\text{If} \quad x \text{ is } X_i \quad \text{and} \quad y \text{ is } Y_j \quad \Rightarrow \quad u \text{ is } U_{i,j}. \tag{4.7}$$

The output for particular input $(x_o(t), y_o(t))$ is

$$\begin{aligned} u &= \frac{\sum_{i=1}^{m}\sum_{j=1}^{m}\mu_{x,i}(t)\mu_{y,j}(t)u_{i,j}}{\sum_{i=1}^{m}\sum_{j=1}^{m}\mu_{x,i}(t)\mu_{y,j}(t)} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m}\mu_{x,i}(t)\mu_{y,j}(t)u_{i,j} \qquad \text{as } \sum_{i=1}^{m}\sum_{j=1}^{m}\mu_{x,i}(t)\mu_{y,j}(t) = 1 \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m}\mu_{i,j}(t)u_{i,j} \end{aligned} \tag{4.8}$$

where,

$$\mu_{i,j}(t) = \mu_{x,i}(t)\mu_{y,j}(t).$$

Writing eqn. (4.8) in matrix form, one has

$$U = \Phi\Theta \tag{4.9}$$

where,

$$U = \begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(h) \end{bmatrix}, \Phi = \begin{bmatrix} \mu_{1,1}(1) & \mu_{1,2}(1) & \cdots & \mu_{m,m}(1) \\ \mu_{1,1}(2) & \mu_{1,2}(2) & \cdots & \mu_{m,m}(2) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{1,1}(h) & \mu_{1,2}(h) & \cdots & \mu_{m,m}(h) \end{bmatrix}, \Theta = \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{m,m} \end{bmatrix}.$$

In eqn. (4.9), the parameters to be estimated, Θ, consists of $u_{i,j}$, for $i, j = 1, 2, \ldots, m$. They can be determined by LSE provided that the columns of Φ are linearly independent. Since the sum of membership degree fired for a particular input is equal to one, the linear equation of $\mu_{i,j}$,

$$\sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} * \mu_{i,j} = 1 \tag{4.10}$$

has the unique solution of $c_{1,1} = c_{1,2} = \cdots = c_{m,m} = 1$. The uniqueness of the solution can be observed from eqn. (4.10) that if the input values are selected such that,

$$x(t) = x_{i_o}, \qquad y(t) = y_{j_o}$$

where, x_{i_o} and y_{j_o} are the values such that only the i_oth linguistic term of x and the j_oth linguistic term of y, respectively, is fired with unity degree. That yields,

$$\mu_{i_o,j_o} = 1, \qquad \mu_{i,j} = 0 \qquad \text{for} \quad i \neq i_o, \;\; j \neq j_o \tag{4.11}$$

Putting eqn. (4.11) into eqn. (4.10) yields $c_{io,jo} = 1$. The same can be carried out for other values of $i, j = 1, 2, \ldots, m$. The unique solution for eqn. (4.10) implies that columns of Φ are linearly independent for any kind of normal membership functions summing to 1. The linear independence can be prove by contradiction that: If the columns of Φ is not linear independent, there exist at least one set of $d_{i,j} \neq 0$, $i, j = 1, 2, \ldots, m$, such that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} d_{i,j} * \mu_{i,j} = 0. \tag{4.12}$$

Putting eqn. (4.12) into eqn. (4.10) yields

$$\sum_{i=1}^{m} \sum_{j=1}^{m} (c_{i,j} + d_{i,j}) * \mu_{i,j} = 1$$

which contradicts the uniqueness of solution for eqn. (4.10). Hence, the columns of Φ are linear independent and LSE can be readily carried out. In fact, some existing adaptive fuzzy controllers [40] are based on identification of the linguistic model.

Example 4.2: Consider the two input linguistic fuzzy system shown in eqn. (4.7). For each input variable, two linguistic terms are defined and placed at -1 and 1. Triangular membership functions, as shown in Fig. 4.1(a), are used within the range of interest (-1,1).

Using the same input data as example 4.1, the matrix of $\Phi = (\mu_{1,1}(i) \;\; \mu_{1,2}(i) \;\; \mu_{2,1}(i) \;\; \mu_{2,2}(i))$, where $i = 1, 2, \ldots, 12$, becomes

$$\Phi = \begin{bmatrix} 0.039 & 0.060 & 0.575 & 0.114 & 0.313 & 0.696 & 0.014 & 0.301 & 0.188 & 0.306 & 0.714 & 0.308 \\ 0.486 & 0.035 & 0.075 & 0.301 & 0.407 & 0.214 & 0.031 & 0.534 & 0.142 & 0.325 & 0.046 & 0.072 \\ 0.036 & 0.570 & 0.310 & 0.161 & 0.122 & 0.069 & 0.291 & 0.059 & 0.382 & 0.179 & 0.226 & 0.502 \\ 0.439 & 0.335 & 0.040 & 0.424 & 0.158 & 0.021 & 0.663 & 0.106 & 0.288 & 0.190 & 0.014 & 0.118 \end{bmatrix}^T$$

It is seen that the rank of $\Phi = 4$. The matrix, $\Phi^T\Phi$, is invertible and the inverse matrix

$$(\Phi^T\Phi)^{-1} = \begin{bmatrix} 1.511 & -1.300 & -1.484 & 1.180 \\ -1.300 & 2.734 & 1.352 & -1.967 \\ -1.484 & 1.352 & 3.209 & -2.341 \\ 1.180 & -1.967 & -2.341 & 3.134 \end{bmatrix}$$

LSE identification can hence be conducted for this case.

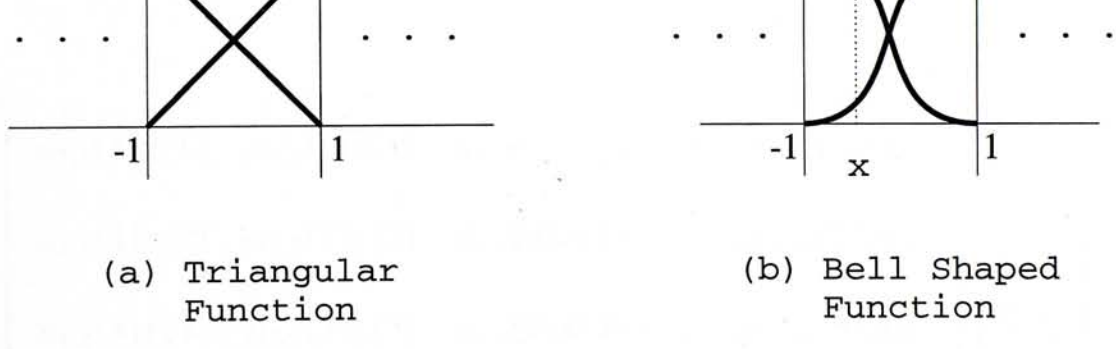


Fig 4.1 The Membership Functions

4.3.2 TSK Model

Consider a linear TSK fuzzy system with two inputs. Each input have m linguistic terms. The general rule is:

$$\text{If} \quad x \text{ is } X_i \quad \text{and} \quad y \text{ is } Y_j \quad \Rightarrow \quad u = \alpha_{i,j}x + \beta_{i,j}y. \tag{4.13}$$

The overall output now becomes

$$\begin{aligned} u &= \sum_{i=1}^{m}\sum_{j=1}^{m}\mu_{i,j}(\alpha_{i,j}x + \beta_{i,j}y) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m}(\alpha_{i,j}(\mu_{i,j}x) + \beta_{i,j}(\mu_{i,j}y)). \end{aligned} \tag{4.14}$$

The parameters to be estimated are $\alpha_{i,j}$ and $\beta_{i,j}$, $i, j = 1, 2, \ldots, m$. The corresponding Φ and Θ are:

$$\Phi = \begin{bmatrix} \mu_{1,1}(1)x(1) & \mu_{1,1}(1)y(1) & \mu_{1,2}(1)x(1) & \cdots & \mu_{m,m}(1)y(1) \\ \mu_{1,1}(2)x(2) & \mu_{1,1}(2)y(2) & \mu_{1,2}(2)x(2) & \cdots & \mu_{m,m}(2)y(2) \\ \mu_{1,1}(3)x(3) & \mu_{1,1}(3)y(3) & \mu_{1,2}(3)x(3) & \cdots & \mu_{m,m}(3)y(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{1,1}(h)x(h) & \mu_{1,1}(h)y(h) & \mu_{1,2}(h)x(h) & \cdots & \mu_{m,m}(h)y(h) \end{bmatrix}, \Theta = \begin{bmatrix} \alpha_{1,1} \\ \beta_{1,1} \\ \alpha_{1,2} \\ \vdots \\ \beta_{m,m} \end{bmatrix}.$$

Eqn. (4.14) has one linearly dependent variable for triangular membership function. It can be shown by considering the homogeneous linear system:

$$\sum_{i=1}^{m}\sum_{j=1}^{m}(c_{i,j}(\mu_{i,j}x) + d_{i,j}(\mu_{i,j}y)) = 0 \tag{4.15}$$

where, $c_{i,j}$, $d_{i,j}$ are arbitrary constants.

For a particular input (x_o, y_o), let fuzzy rules (i,j), $(i,j+1)$, $(i+1,j)$ and $(i+1,j+1)$ be fired with membership degree $\mu_{i,j}, \mu_{i,j+1}, \mu_{i+1,j}, \mu_{i+1,j+1}$, respectively. Remaining rules are not fired. Eqn. (4.15) can be written as

$$\begin{aligned}&\mu_{i,j}(c_{i,j}x_o + d_{i,j}y_o) + \mu_{i,j+1}(c_{i,j+1}x_o + d_{i,j+1}y_o) + \mu_{i+1,j}(c_{i+1,j}x_o\\ &+d_{i+1,j}y_o) + \mu_{i+1,j+1}(c_{i+1,j+1}x_o + d_{i+1,j+1}y_o) = 0\end{aligned} \tag{4.16}$$

When triangular membership functions are utilized, i.e. $\mu_{x,i} = \frac{x_{i+1}-x}{x_{i+1}-x_i}$, $\mu_{y,j} = \frac{y_{j+1}-y}{y_{j+1}-y_j}$, $\mu_{x,i+1} = 1 - \mu_{x,i}$ and $\mu_{y,j+1} = 1 - \mu_{y,j}$, and eqn. (4.16) becomes

$$\begin{aligned}&\frac{x_{i+1}-x_o}{x_{i+1}-x_i}\frac{y_{j+1}-y_o}{y_{j+1}-y_j}(c_{i,j}x_o + d_{i,j}y_o) + \frac{x_{i+1}-x_o}{x_{i+1}-x_i}\frac{y_o-y_j}{y_{j+1}-y_j}(c_{i,j+1}x_o\\ &+d_{i,j+1}y_o) + \frac{x_o-x_i}{x_{i+1}-x_i}\frac{y_{j+1}-y_o}{y_{j+1}-y_j}(c_{i+1,j}x_o + d_{i+1,j}y_o)\\ &+\frac{x_o-x_i}{x_{i+1}-x_i}\frac{y_o-y_{j+1}}{y_{j+1}-y_j}(c_{i+1,j+1}x_o + d_{i+1,j+1}y_o) = 0\end{aligned} \tag{4.17}$$

which yields,

$$(x_{i+1} - x_o)(y_{j+1} - y_o)(c_{i,j}x_o + d_{i,j}y_o) + (x_{i+1} - x_o)(y_o - y_j)(c_{i,j+1}x_o$$
$$+d_{i,j+1}y_o) + (x_o - x_i)(y_{j+1} - y_o)(c_{i+1,j}x_o + d_{i+1,j}y_o)$$
$$+(x_o - x_i)(y_o - y_j)(c_{i+1,j+1}x_o + d_{i+1,j+1}y_o) = 0 \qquad (4.18)$$

Eqn. (4.18) can be solved by *MathCAD* and yields one set of non-trivial solution for eqn. (4.15),

$$c_{i,j} = c_{i+1,j} = y_j,$$
$$c_{i,j+1} = c_{i+1,j+1} = y_{j+1},$$
$$d_{i,j} = d_{i,j+1} = -x_i,$$
$$d_{i+1,j} = d_{i+1,j+1} = -x_{i+1}. \qquad (4.19)$$

Generalizing, it can be shown that the solutions of $c_{i,j}$ and $d_{i,j}$'s to eqn. (4.15) are:

$$c_{\square,j} = y_j, \qquad d_{i,\square} = -x_i \qquad (4.20)$$

for all $i, j = 1, \cdots, m$.

The above result shows that, for triangular membership function, one column of the resultant matrix Φ is linearly dependent on the others. In order to conduct LSE, the system must be reformulated. However this is not straightforward especially when the number of input is large. One alternative is to use membership functions other than triangular shape, such as bell shape function. In fact, identification on TSK model reported in [40] actually adopted bell shaped membership function.

Example 4.3: Consider the two input linguistic fuzzy system shown in eqn. (4.7). For each input variable, two linguistic terms are defined and placed at -1 and 1.

Consider first the case where triangular membership, as shown in Fig. 4.1(a), is utilized.

Using the same data as examples 4.1, the matrix $\Phi = (\mu_{1,1}(i)x(i) \quad \mu_{1,1}y(i) \quad \mu_{1,2}(i)x(i) \quad \mu_{1,2}y(i) \quad \mu_{2,1}(i)x(i) \quad \mu_{2,1}y(i) \quad \mu_{2,2}(i)x(i) \quad \mu_{2,2}y(i))$, where $i = 1, 2, \ldots, m$, is

$$
\Phi = \begin{bmatrix}
0.002 & -0.034 & 0.024 & -0.413 & 0.002 & -0.030 & 0.022 & -0.374 \\
-0.049 & 0.016 & -0.029 & 0.009 & -0.462 & 0.148 & -0.271 & 0.087 \\
0.173 & 0.443 & 0.022 & 0.058 & 0.093 & 0.239 & 0.012 & 0.031 \\
-0.019 & -0.051 & -0.051 & -0.135 & -0.027 & -0.072 & -0.072 & -0.191 \\
0.138 & -0.041 & 0.179 & -0.053 & 0.054 & -0.016 & 0.070 & -0.021 \\
0.571 & 0.369 & 0.175 & 0.113 & 0.057 & 0.037 & 0.017 & 0.011 \\
-0.013 & -0.005 & -0.029 & -0.012 & -0.265 & -0.114 & -0.604 & -0.259 \\
0.201 & -0.084 & 0.358 & -0.150 & 0.040 & -0.017 & 0.071 & -0.030 \\
-0.064 & 0.026 & -0.048 & 0.020 & -0.130 & 0.054 & -0.098 & 0.040 \\
0.079 & -0.009 & 0.084 & -0.010 & 0.047 & -0.005 & 0.050 & -0.006 \\
0.372 & 0.629 & 0.024 & 0.040 & 0.117 & 0.199 & 0.008 & 0.013 \\
-0.074 & 0.191 & -0.017 & 0.045 & -0.121 & 0.311 & -0.028 & 0.073
\end{bmatrix}
$$

The rank of $\Phi = 7$. The system has one linearly dependent variable. The matrix $\Phi^T\Phi$ is non-invertible.

If the bell shaped membership function, as shown in Fig. 4.2(b), is used, one has

$$\Phi = \begin{bmatrix} 0.001 & -0.005 & 0.027 & -0.461 & 0.001 & -0.004 & 0.022 & -0.379 \\ -0.011 & 0.003 & -0.004 & 0.001 & -0.577 & 0.185 & -0.218 & 0.070 \\ 0.221 & 0.566 & 0.006 & 0.015 & 0.071 & 0.184 & 0.002 & 0.005 \\ -0.009 & -0.023 & -0.049 & -0.131 & -0.017 & -0.045 & -0.094 & -0.250 \\ 0.140 & -0.042 & 0.231 & -0.068 & 0.026 & -0.008 & 0.043 & -0.013 \\ 0.717 & 0.464 & 0.089 & 0.057 & 0.012 & 0.008 & 0.002 & 0.001 \\ -0.001 & -0.001 & -0.003 & -0.001 & -0.168 & -0.072 & -0.737 & -0.316 \\ 0.164 & -0.068 & 0.469 & -0.196 & 0.009 & -0.004 & 0.027 & -0.011 \\ -0.047 & 0.019 & -0.027 & 0.011 & -0.167 & 0.069 & -0.098 & 0.041 \\ 0.089 & -0.010 & 0.100 & -0.012 & 0.033 & -0.004 & 0.037 & -0.004 \\ 0.457 & 0.773 & 0.003 & 0.006 & 0.059 & 0.101 & 0.001 & 0.001 \\ -0.064 & 0.166 & -0.005 & 0.013 & -0.158 & 0.409 & -0.012 & 0.032 \end{bmatrix}$$

and the rank of $\Phi = 8$. The system attains full rank. The matrix, $\Phi^T\Phi$, is invertible and

$(\Phi^T\Phi)^{-1} =$

$$\begin{bmatrix} 20.47 & -16.97 & -24.34 & -37.98 & 19.94 & 16.79 & -23.59 & 39.73 \\ -16.97 & 15.54 & 19.69 & 29.99 & -17.54 & -16.25 & 19.51 & -31.50 \\ -24.34 & 19.69 & 37.69 & 62.34 & -29.08 & -19.48 & 36.62 & -64.86 \\ -37.98 & 29.99 & 62.34 & 126.12 & -54.48 & -29.65 & 72.62 & -131.81 \\ 19.94 & -17.54 & -29.08 & -54.48 & 31.51 & 20.92 & -35.58 & 58.45 \\ 16.79 & -16.25 & -19.48 & -29.65 & 20.92 & 23.04 & -20.81 & 30.78 \\ -23.59 & 19.51 & 36.62 & 72.62 & -35.58 & -20.81 & 46.33 & -78.47 \\ 39.73 & -31.50 & -64.86 & -131.81 & 58.45 & 30.78 & -78.47 & 142.83 \end{bmatrix}$$

LSE identification can hence be readily conducted.

4.3.3 Decomposable System

Parameters for a decomposable system can be approximated from the identified general fuzzy system, or they can be estimated directly from input-output data. Consider the case for two inputs and each having m linguistic terms. The general rules for the subsystems are:

$$\text{If} \quad x \text{ is } X_i \quad \Rightarrow \quad u_x \text{ is } U_{xi} \tag{4.21}$$

and,

$$\text{If} \quad y \text{ is } Y_j \quad \Rightarrow \quad u_y \text{ is } U_{yj} \tag{4.22}$$

The overall output for the input (x,y) is then

$$u = u_x + u_y = \sum_{i=1}^{m} \mu_{x,i} u_{xi} + \sum_{j=1}^{m} \mu_{yj} u_{yj}. \tag{4.23}$$

The parameters to be estimated are u_{xi} and u_{yj}, $i, j = 1, 2, \ldots, m$. Corresponding Θ and Φ are expressed as:

$$\Theta = \begin{bmatrix} u_{1,1} \\ u_{1,2} \\ \vdots \\ u_{m,m} \end{bmatrix}, \Phi = [\Phi_X \quad \Phi_Y],$$

$$\Phi_X = \begin{bmatrix} \mu_{x,1}(1) & \mu_{x,2}(1) & \cdots & \mu_{x,m}(1) \\ \mu_{x,1}(2) & \mu_{x,2}(2) & \cdots & \mu_{x,m}(2) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{x,1}(h) & \mu_{x,2}(h) & \cdots & \mu_{x,m}(h) \end{bmatrix},$$

$$\Phi_Y = \begin{bmatrix} \mu_{y,1}(1) & \mu_{y,2}(1) & \cdots & \mu_{y,m}(1) \\ \mu_{y,1}(2) & \mu_{y,2}(2) & \cdots & \mu_{y,m}(2) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{y,1}(h) & \mu_{y,2}(h) & \cdots & \mu_{y,m}(h) \end{bmatrix}.$$

Eqn. 4.23 has one linearly dependent variable regarding the shape of membership function applied. Since, x and y are independent variables, columns of Φ_X and Φ_Y are linearly independent, respectively. However, as the sum of membership degree for each fuzzy variable equals to 1

$$\sum_{i=1}^{m} \mu_{x,i} = \sum_{j=1}^{m} \mu_{y,j} = 1$$

There exist one linearly dependent column in Φ. Thus, one of the variables of Θ can be regarded as linearly dependent on the others. This is the same problem encountered in Chapter 3 regarding matrix $\mathcal{D}$. The problem can be easily reformulated to eliminate the dependency, see chapter 3 or [35].

Example 4.4: Consider the decomposable fuzzy system with two fuzzy subsystems as shown in eqn. (4.22). For each input variable, two linguistic terms are defined and placed at -1 and 1. Triangular membership functions, as shown in Fig. 4.1(a), are used in this case.

The parameter vector Θ and matrix Φ are given as: $\Theta = (u_{x,1} \;\; u_{x,2} \;\; u_{y,1} \;\; u_{y,2})^T$, $\Phi = (\mu_{x,1}(i) \;\; \mu_{x,2}(i) \;\; \mu_{y,1}(i) \;\; \mu_{y,2}(i))$, where $i = 1, 2, \ldots, h$. As discussed above, there is one linearly dependent variable in Φ. The problem can be reformulated to eliminate the dependency by using $\mu_{y,2}(i) = \mu_{x,1}(i) + \mu_{x,2}(i) - \mu_{y,1}(i)$ and defining new parameter vector Θ' and matrix Φ' as

$$\Theta' = \begin{bmatrix} u'_{x,1} \\ u'_{x,2} \\ u'_{y,1} \end{bmatrix} = \begin{bmatrix} u_{x,1} - u_{y,2} \\ u_{x,2} - u_{y,2} \\ u_{y,1} - u_{y,2} \end{bmatrix}$$

$$\Phi' = \begin{bmatrix} \mu_{x,1}(1) & \mu_{x,2}(1) & \mu_{y,1}(1) \\ \mu_{x,1}(2) & \mu_{x,2}(2) & \mu_{y,1}(2) \\ \vdots & \vdots & \vdots \\ \mu_{x,1}(h) & \mu_{x,2}(h) & \mu_{y,1}(h) \end{bmatrix}.$$

Equation (4.9) hence becomes $U = \Phi'\Theta'$. Using the same data as examples 4.1, the matrix $\Phi' = (\mu_{x,1}(i) \;\; \mu_{x,2}(i) \;\; \mu_{y,1}(i))$, where $i = 1, 1, \ldots, 12$, is

$$\Phi' = \begin{bmatrix} 0.525 & 0.095 & 0.650 & 0.415 & 0.720 & 0.9100 & 0.045 & 0.835 & 0.330 & 0.630 & 0.760 & 0.3800 \\ 0.475 & 0.905 & 0.350 & 0.585 & 0.280 & 0.0900 & 0.955 & 0.165 & 0.670 & 0.370 & 0.240 & 0.6200 \\ 0.075 & 0.630 & 0.885 & 0.275 & 0.435 & 0.7650 & 0.305 & 0.360 & 0.570 & 0.485 & 0.940 & 0.8100 \end{bmatrix}^T$$

It is seen that the rank of $\Phi' = 3$. The matrix, $\Phi'^T\Phi'$, is invertible and the inverse matrix

$$(\Phi'^T\Phi')^{-1} = \begin{bmatrix} 0.92 & 0.16 & -0.87 \\ 0.16 & 0.64 & -0.56 \\ -0.87 & -0.56 & 1.33 \end{bmatrix}.$$

LSE can be readily carried out and Θ' can hence be determined with $u_{y,2}$ as free parameter which can be set any convenient value.

4.3.4 Comparative Case Study

Consider a Van der Pol equation:

$$\ddot{x} = -x + \zeta(1 - x)\dot{x} \tag{4.24}$$

with $\zeta = 1$. The range of interest is [-5,5] for both x and $\dot{x}$. Assuming that five linguistic terms, denoted by **NL**(negative large), **NS**(negative small), **ZO**(zero), **PS**(positive small) and **PL**(positive), are used. They are equally divided within the range of interest, i.e. **NL** = -5, **NS** = -2.5, **ZO** = 0, **PS** = 2.5 and **PL** = 5. As the linear TSK system exhibits linear dependency with triangular membership functions, bell shape membership functions, as in Fig. 4.1(b), are adopted for the TSK model. For other systems, triangular membership functions are used in the identification process. For the decomposable model, linear dependency is eliminated by reformulation as in example 4.4.

The system is excited with 2000 random inputs. The identification results for different models are tabulated in Table 4.1, Table 4.2 and Table 4.3. Profiles of the original nonlinear system and the identified models are shown in Fig. 4.2. Outputs of the fuzzy models are inferred by triangular membership function including the TSK model (though its parameters are identified via bell-shape functions). It can be observed that the output profiles for both linguistic (Fig. 4.2(b)) and TSK (Fig. 4.2(c)) model are quite similar to the original one while that of the decomposable model (Fig. 4.2(d)) is not. This is due to the fact that decomposable approximation does not incorporate cross product terms whereas the Van der Pol equation has heavily weighted cross products. It is hence prudent to check the nature of the system under investigation before deciding upon the system structure.

		x				
		NL	NS	ZO	PS	PL
$\dot{x}$	NL	119.93	62.61	5.07	-52.49	-108.85
	NS	22.78	12.96	2.49	-8.19	-19.00
	ZO	-10.36	-5.11	-0.03	5.15	10.36
	PS	18.65	8.26	-2.50	-13.05	-22.51
	PL	110.12	52.36	-4.99	-62.31	-120.75

Table 4.1: Identification result on linguistic model

4.4 Fuzzy Regional System Identification

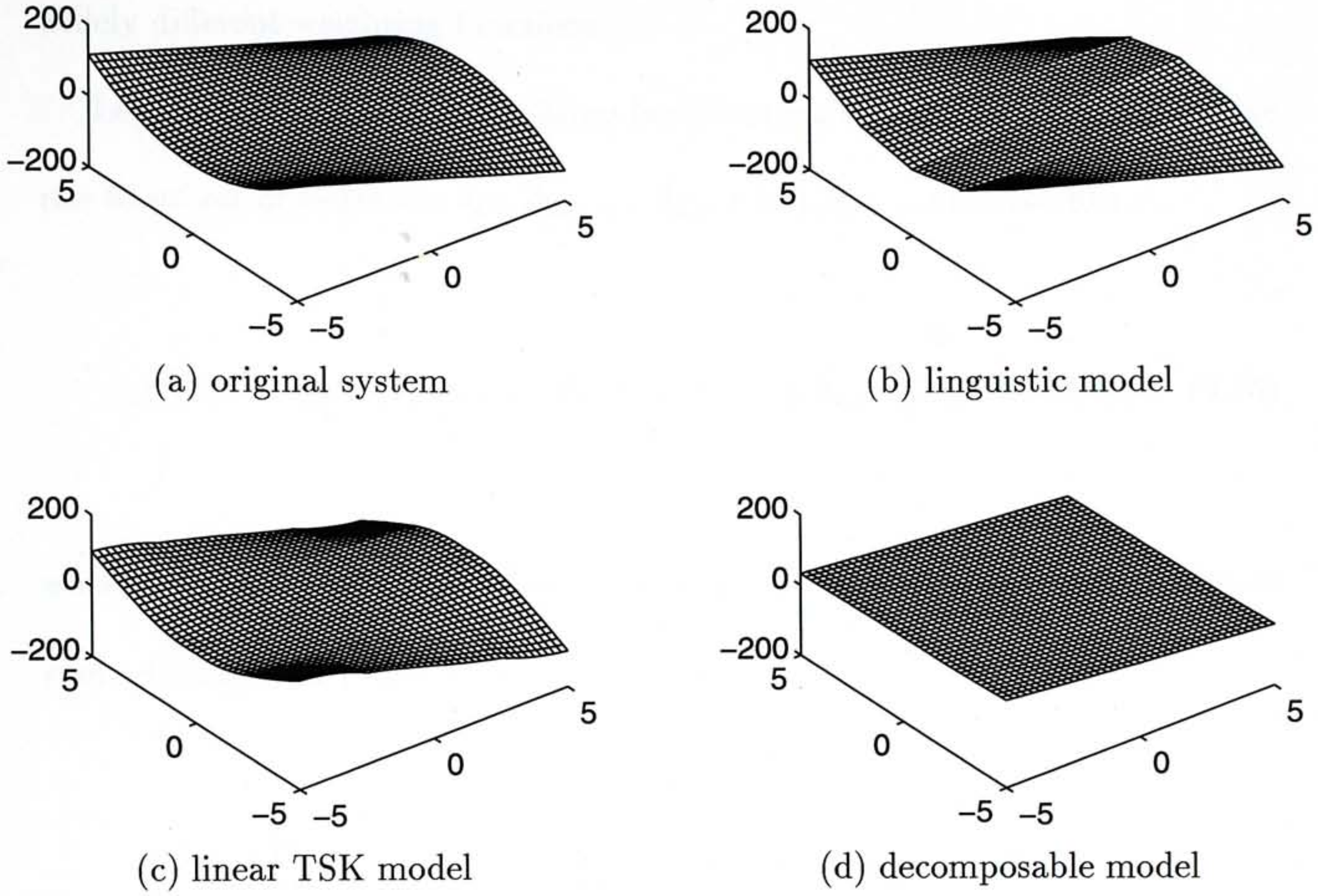
This section, develops a regional fuzzy system identification technique whereby membership functions would act as weighting factors in a localized identification problem. Consider first a conventional weighted least-squares identification problem. Weighting function $w(i)$ can be incorporated into the regression function eqn. (4.2) as

		x				
		NL	NS	ZO	PS	PL
$\dot{x}$	NL	$-19.68x$ $-2.07\dot{x}$	$-5.47x$ $-12.46\dot{x}$	$-1.07x$ $-14.80\dot{x}$	$3.44x$ $-12.49\dot{x}$	$17.08x$ $-2.48\dot{x}$
	NS	$-16.71x$ $+2.89\dot{x}$	$-6.26x$ $+0.41\dot{x}$	$-0.92x$ $-0.88\dot{x}$	$3.84x$ $-0.36\dot{x}$	$14.08x$ $+2.77\dot{x}$
	ZO	$-1.39x$ $+0.63\dot{x}$	$-1.39x$ $+1.05\dot{x}$	$-0.99x$ $+1.20\dot{x}$	$-1.53x$ $+1.04\dot{x}$	$-1.14x$ $+0.62\dot{x}$
	PS	$13.49x$ $+2.52\dot{x}$	$3.98x$ $-0.11\dot{x}$	$-0.97x$ $-1.04\dot{x}$	$-6.39x$ $+0.58\dot{x}$	$-17.34x$ $+3.20\dot{x}$
	PL	$18.90x$ $-1.28\dot{x}$	$3.94x$ $-11.39\dot{x}$	$-1.07x$ $-14.29\dot{x}$	$-4.98x$ $-12.98\dot{x}$	$-20.75x$ $-0.67\dot{x}$

Table 4.2: Identification result on linear TSK model

x				
NL	NS	ZO	PS	PL
7.57	1.02	0.00	-2.84	-4.51
$\dot{x}$				
NL	NS	ZO	PS	PL
36.59	18.05	0.00	-19.26	-29.43

Table 4.3: Identification result on linguistic model



(a) original system

(b) linguistic model

(c) linear TSK model

(d) decomposable model

Fig 4.2 Output Profile of the Original System and Various Fuzzy Model Identification

$$w(i)u(i) = w(i)(\theta_1 x_1(i) + \theta_2 x_2(i) + \ldots + \theta_n x_n(i) + e(i)) \tag{4.25}$$

for $i = 1, 2, \ldots, h$. The weighting function $w(i)$ denotes the importance of a particular set of input-output data. It marks the degree of significant of the data in concern. If two weighting functions are similar, the parameters identified would be similar. However, the parameters identified may differ greatly for two widely different weighting functions.

Let there be m' sets of weighting functions $w_k(i), i = 1, 2, \ldots, h$ which give rise to m' set of estimates $\hat{\theta}_{k,1}, \hat{\theta}_{k,2}, \ldots, \hat{\theta}_{k,n}, k = 1, 2, \ldots, m'$. One has

$$w_k(i)u(i) = w_k(i)(\hat{\theta}_{k,1}x_1(i) + \hat{\theta}_{k,2}x_2(i) + \ldots + \hat{\theta}_{k,n}x_n(i)) + w_k(i)e_k(i) \quad (4.26)$$

where $\hat{\theta}_{k,i}$ results in "smallest" errors $e_k(i)$ in the sense of weighted square error sum. Adding eqn. (4.26) for $k = 1, 2, \ldots, m'$.

$$\begin{aligned} u(i)\sum_{k=1}^{m'} w_k(i) &= x_1(i)\sum_{k=1}^{m'} w_k(i)\theta_{k,1} + x_2(i)\sum_{k=1}^{m'} w_k(i)\theta_{k,2} \\ &\quad + \ldots + x_n(i)\sum_{k=1}^{m'} w_k(i)\theta_{k,n} + \sum_{k=1}^{m'} w_k(i)e_k(i) \end{aligned}$$

or, omitting the error terms,

$$\begin{aligned} u(i) &= x_1(i)\frac{\sum_{k=1}^{m'} w_k(i)\hat{\theta}_{k,1}}{\sum_{k=1}^{m'} w_k(i)} + x_2(i)\frac{\sum_{k=1}^{m'} w_k(i)\hat{\theta}_{k,2}}{\sum_{k=1}^{m'} w_k(i)} \\ &\quad + \ldots + x_n(i)\frac{\sum_{k=1}^{m'} w_k(i)\hat{\theta}_{k,n}}{\sum_{k=1}^{m'} w_k(i)} \end{aligned} \quad (4.27)$$

If the weighting functions are chosen as the product of membership functions in the fuzzy rules, eqn. (4.27) is equivalent to a TSK fuzzy inference with linear terms. The number m' hence equals the number of fuzzy rules in this case. The values $\hat{\theta}_{k,1}, \ldots, \hat{\theta}_{k,n}$ are determined with eqn. (4.26) within regional domains as defined by the membership functions in concern. In this situation, the process of eqn. (4.26) involves the errors within individual region instead of the global errors of the whole system. The fuzzy regional identification utilizes the information of neighbourhood regions to the extent as according to the shape of membership functions. If the membership function in eqn. (4.26) is rectangular, crisp set, the process would be equivalent to conventional regional identification analysis. Moreover, the regional analysis considers different regions of the whole system individually. In case that some regions are not sampled, the corresponding region would be empty and no action will be taken there. Hence, regional analysis has the ability of accommodating the situation where some regions are empty.

In general, the number of fuzzy rules required is unknown ahead of time. The global analysis approach can start with a pre-defined set of membership functions which seem reasonable and then gradually insert more rules and redefine the membership functions if the result of identification is not closed to the original system. This requires solving the regression equations for the fuzzy system repeatedly. An alternative is to divide the fuzzy system into subsystems and conduct regional LSE of the subsystems.

4.4.1 Case Study

Using the same set of data as generated before in section 4.3.4, the Van der Pol equation (4.24) is identified by linguistic model with rectangular, crisp membership function. The identification results, which are equivalent to conventional regional identification analysis, are tabulated in Table 4.4. They are very close to that given in Table 4.1. Fig. 4.3 presents the outputs as inferred by triangular membership functions using regional identification results and global LSE estimation results (same as Fig. 4.2(b)). It can be seen from Fig. 4.3 that the two output profiles are very closed to each other.

Regional identification process with triangular membership function is then performed and compare to that generated from global identification. The results are tabulated in Table 4.5 which is, again, very close to that generated from global identification of Table 4.1. Furthermore, Fig. 4.4. and Table 4.6 shows regional identification in case when certain regions are not sampled.

4.5 Recursive Estimation

Recursive algorithms for weighted least-squares estimation can be directly applied to regional identification of fuzzy systems.

		x				
		NL	NS	ZO	PS	PL
$\dot{x}$	NL	119.73	62.67	4.93	-52.17	-108.62
	NS	23.55	12.94	2.50	-8.04	-18.22
	ZO	-9.94	-5.33	0.04	5.25	9.81
	PS	17.88	7.96	-2.34	-13.09	-23.29
	PL	109.51	52.81	-5.16	-62.22	-121.72

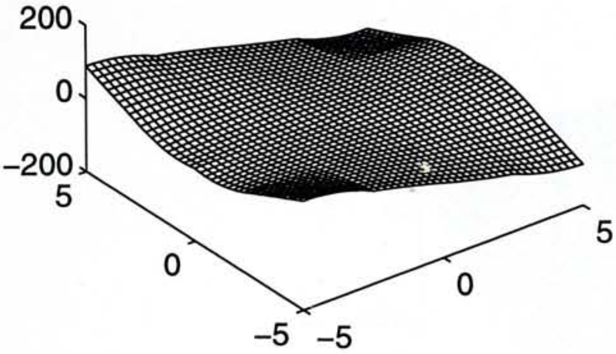
Table 4.4: Identification Result on Linguistic Model with Rectangular Membership Function

		x				
		NL	NS	ZO	PS	PL
$\dot{x}$	NL	110.54	58.43	4.83	-50.17	-100.07
	NS	25.25	14.31	2.66	-8.68	-19.41
	ZO	-8.26	-3.89	-0.02	3.17	9.16
	PS	20.88	9.63	-3.04	-15.38	-25.29
	PL	105.24	49.79	-4.62	-58.11	-114.38

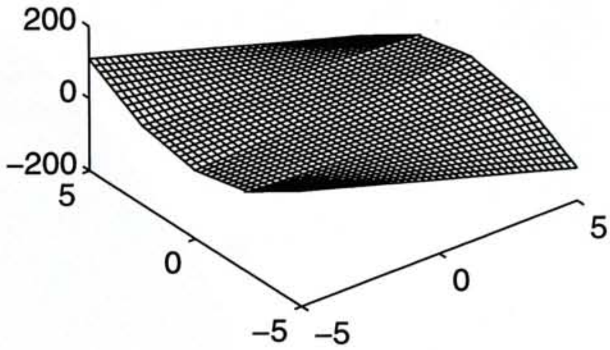
Table 4.5: Identification Result Based on Fuzzy Regional Analysis

		x				
		NL	NS	ZO	PS	PL
$\dot{x}$	NL	109.89	59.02	5.01	–	–
	NS	24.47	14.15	2.38	–	–
	ZO	-9.13	-3.46	-0.02	3.89	8.79
	PS	–	–	-2.73	-15.14	-24.88
	PL	–	–	-4.80	-59.33	-116.36

Table 4.6: Identification Result Based on Fuzzy Regional Analysis



(a)



(b)

Fig. 4.3 Output Profile of Identification Result with
(a) Rectangular Membership Function and
(b) Triangular Membership Function

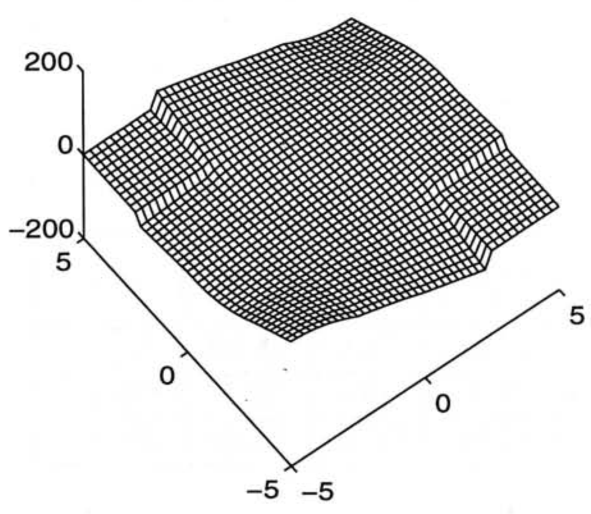


Fig. 4.4 Output Profile of Identification Result with Missing Rules

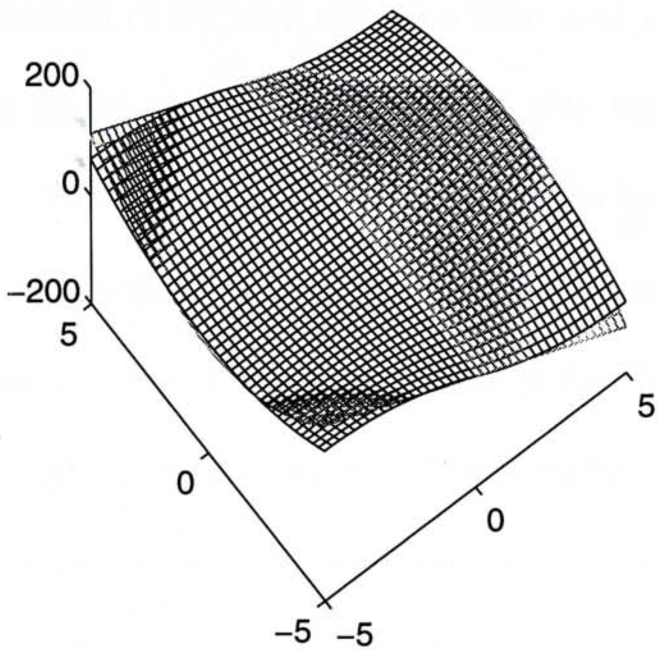


Fig 4.5 Output Profile of the Original System Vs Fuzzy System with Three Linguistic Terms

		x		
		NL	ZO	PL
$\dot{x}$	NL	60.83	2.87	-51.22
	ZO	3.32	0.06	-4.17
	PL	48.07	-3.32	-59.75

Table 4.7: Fuzzy Identification For Three Linguistic Terms

4.5.1 Case Study

Here, the regional approach of section 4.4 is conducted recursively to identify the Van der Pol equation (4.24) with the same 2000 data utilized in case study 4.3.4. A linguistic model of two fuzzy variables with triangular membership functions are used. Three linguistic terms are assumed for each fuzzy variable first. The results are tabulated in Table 4.7 and the output profile is shown in Fig. 4.5. It is revealed in Fig. 4.5 that the profile of the identified fuzzy system is not close to that of the original system. This is mainly due to the fact that 3 linguistic terms is not fine enough to approximate the original system. With two more linguistic terms inserted for each fuzzy variable, the new system as identified regionally and recursively then become very close to that in Table 4.5 which was obtained by regional analysis in batch process.

4.6 Conclusion

In this chapter, fuzzy identification is formulated as linear regression problem. It is shown that some choice of membership functions may result in linearly dependency. Membership functions must hence be chosen carefully. Fuzzy regional identification, which results in membership function acting as weighting function in a localized estimation problem, is also discussed. The regional problem minimizes errors within individual domain instead of the global domain of the full system. The approach does not have the problem of linearly dependent variables, as each subsystems handles input data individually. Moreover, the regional approach is applicable even in the case that some regions are not sampled.

Chapter 5

Performance-Based Fuzzy Gain Controller

5.1 Introduction

Advantage of fuzzy control lies in its ability to mimic human's experience and knowledge via incorporation of heuristic decision rules [41, 42, 1, 8]. However, skepticism to fuzzy control lingers as it constitutes a non-mathematical algorithm [13, 43] which renders analysis of system performance and stability difficulty as compared to conventional feedback. In addition, there is no systematic design method for fuzzy controller, much efforts must be spent on calibration of various quantities to enhance performance.

Standard fuzzy control involves using a fuzzy inference to produce the control input to a plant. In this section a new approach in which a fuzzy inference system is utilized to produce the "gain" value of the input is investigated. This fuzzy gain approach is more in line with traditional feedback techniques in which a controller is characterized by its input/output descriptions of gain or transfer functions. A simple fuzzy inference mechanism with a single fuzzy input variable and triangular-shaped membership functions is presented here to illustrate the concept. This is with understanding that a general fuzzy controller can actually be decomposed into or approximated by fuzzy subsystems of single input variable.

5.2 Conventional Fuzzy Control

Consider the following fuzzy controller:

$$\begin{aligned}
\text{Rule } 1 &: \text{ If } x \text{ is } X_1 \Rightarrow u \text{ is } U_1, \\
\vdots \quad & \quad\quad \vdots \quad\quad\quad \vdots \\
\text{Rule } m &: \text{ If } x \text{ is } X_m \Rightarrow u \text{ is } U_m.
\end{aligned}$$

where, respectively, triangular and singleton membership functions are assumed for X_i and U_i, see Fig. 5.1. Let X_i, X_{i+1} are fired with strength a and $(1-a)$ for some input x_o. As $a = \frac{x_{i+1}-x_o}{x_{i+1}-x_i}$, the inferred control value is expressed as:

$$u = u_i a + u_{i+1}(1-a) = c_{1i} + c_{2i}x_o \tag{5.1}$$

where, $c_{1i} = \frac{u_i x_{i+1} - u_{i+1} x_i}{x_{i+1}-x_i}$ and $c_{2i} = \frac{u_{i+1}-u_i}{x_{i+1}-xi}$. The control input u is hence in general a non-linear function of x_o, with offsets c_{1i} and varying slope c_{2i}. Eqn. 5.1 can be rewritten as: $u = x_o \alpha(x_o)$, with $\alpha(x_o) = \frac{c_{1i}}{x_o} + c_{2i}$. This shows that conventional fuzzy controller is similar to a proportional gain controller with gain $\alpha(x_o)$. Inherent difficulty in design and analysis of the fuzzy controller results due to the first term of $\alpha(x_o)$ which is inversely proportional to x_o

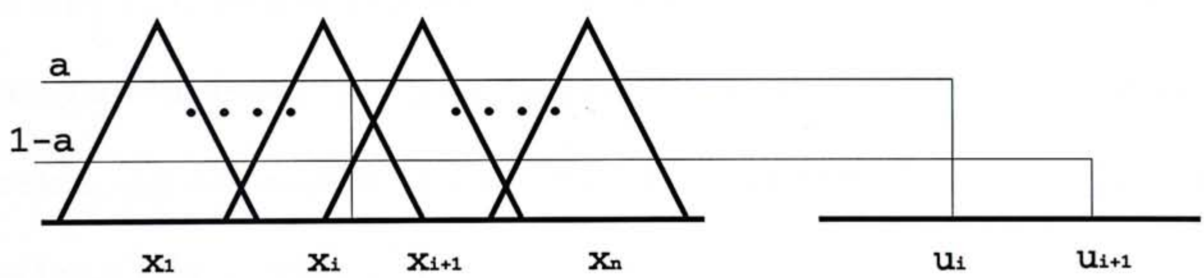


Fig 5.1 Inference Output with Triangular Membership Function

5.3 Fuzzy Gain Control

The above observation motivates the fuzzy gain concept where the inferred output is not the control value but the control gain. The fuzzy gain inference system is:

$$\begin{aligned} &\text{Rule } 1 : \text{If } p \text{ is } P_1 \Rightarrow k \text{ is } K_1, \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &\text{Rule } m : \text{If } p \text{ is } P_m \Rightarrow k \text{ is } K_m. \end{aligned} \tag{5.2}$$

The triangular membership function P_i is located at p_i and the singleton membership function K_i at k_i (similar as those shown in Fig. 5.1). The input variable p is selected to be a certain performance measure of the system. The value k_i is designed to efficiently improve the performance for $p \in P_i$. For the regulator problem, say, one example of p would be the magnitude of the regulated state. The fuzzy gain as inferred by eqn. 5.2 is readily expressed as:

$$k = \frac{\sum_{i=1}^{m} k_i \mu(k_i)}{\sum_{i=1}^{m} \mu(k_i)} \tag{5.3}$$

Given $p = p_o$, the inferred value of k is,

$$k = (\frac{k_i p_{i+1} - k_{i+1} p_i}{p_{i+1} - p_i}) + (\frac{k_{i+1} - k_i}{p_{i+1} - p_i}) p_o \tag{5.4}$$

The control input is then:

$$u = kx_o = (\frac{k_i p_{i+1} - k_{i+1} p_i}{p_{i+1} - pi}) x_o + (\frac{k_{i+1} - k_i}{p_{i+1} - p_i}) x_o p_o \tag{5.5}$$

Unlike the case for conventional fuzzy control, it can be seen that there is now no offset in u. In addition, the gain for $p_o \in (P_i, P_j)$, $i \leq j$, is bounded by $min(k_i, k_{i+1}, \ldots, k_j) \leq k \leq max(k_i, k_{i+1}, \ldots, k_j)$ as shown by eqn. (5.3). The resulting analysis is hence more accommodating to conventional methods.

The present formulation does not confine to proportional gain controller only. The k_i in eqn.(5.2) can actually denote general controller. The main idea here is that the proposed approach introduces the ability for the system to adopt to more efficient controllers as according to its performance during the course of operation. And that instead of jumping to a different gain level upon command, fuzzy inference is adopted to produce a smooth transition from one controller to another. This approach can be view as a generalized version of gain scheduling.

5.4 Design Algorithm

The design algorithm of the performance-base is introduced here.

step 1. Build the system model.

step 2. Design controllers for various degree of damping.

step 3. Choose performance index.

step 4. Set the fuzzy rules which would select the "appropriate" controller as according to performance index.

step 5. Check stability for the performance-based control system. If it is not stable, repeat step 2 for modified controller design.

step 6. If the response is not satisfactory, adjust the fuzzy table or add some intermediate controller by repeating step 2 to 5.

A practical design method for this approach is using the pole-placement method to design a set of controllers, which would have the far poles of the closed loop system staying more or less in the same position with the dominant poles lying on a vertical line in the s-plane as shown in Fig. 5.2.

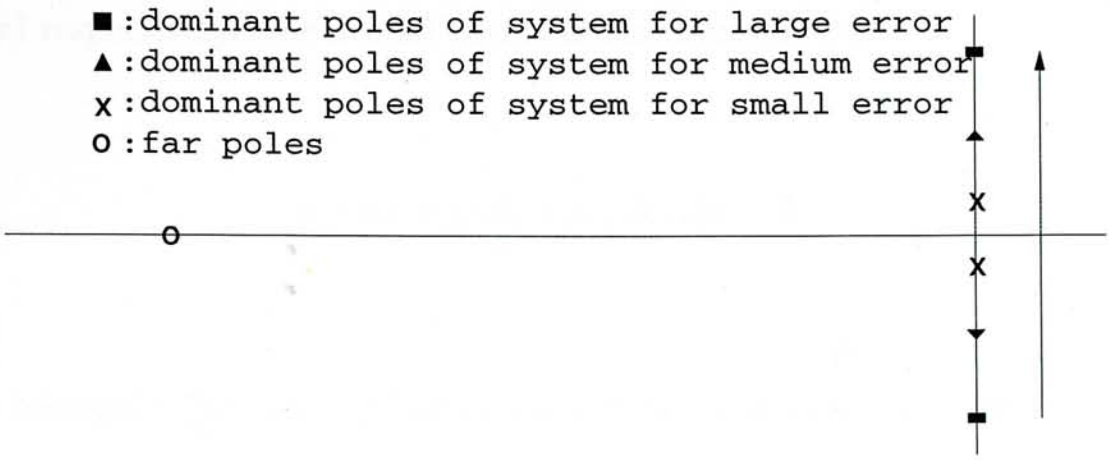


Fig 5.2 Poles Location of the Controller for Different Fuzzy Rules

5.5 Stability Design Approach

In this section, stability of fuzzy inference on second order continuous systems is investigated.

Consider the case where the closed loop system is second order and the set of controllers are designed such that the closed loop poles are located on a vertical line on the left-half of the s-plane (as in Fig. 5.2 without the far poles). For rule i, which is "If x is $X_i \Rightarrow b$ is B_i", the resultant characteristic equation is

$$\ddot{x} + a\dot{x} + b_i x = 0 \tag{5.6}$$

Let the position x be a fuzzy variable with triangular membership functions.

For a particular value of x, rules i and $i+1$ are fired with strength $\mu_i(x)$ and $\mu_{i+1}(x)$ respectively. The overall characteristic equation hence is

$$\ddot{x} + a\dot{x} + (\mu_i b_i + \mu_{i+1} b_{i+1})x = 0. \tag{5.7}$$

Since triangular membership function is adopted, one has,

$$\begin{aligned} & \mu_i b_i + \mu_{i+1} b_{i+1} \\ = & \frac{x_{i+1} b_i - x_i b_{i+1}}{x_{i+1} - x_i} + \frac{b_{i+1} - b_i}{x_{i+1} - x_i} x \qquad (5.8) \\ = & \rho_{i,i+1} + \kappa_{i,i+1} x \qquad (5.9) \end{aligned}$$

If we select b_i, b_{i+1} so that

$$\begin{aligned} \rho_{i,i+1} &> 0 \\ \kappa_{i,i+1} &> 0 \qquad \text{for} \quad x > 0 \qquad (5.10) \\ \kappa_{i,i+1} &< 0 \qquad \text{for} \quad x < 0 \end{aligned}$$

Let $\kappa'_{i,i+1}$ be the magnitude of $\kappa_{i,i+1}$. The overall system equation for x lying between x_i and x_{i+1} can be rewritten as

$$(\ddot{x} + a\dot{x} + \rho_{i,i+1}x) + \kappa'_{i,i+1}|x|x = 0 \tag{5.11}$$

It can be seen that the first three terms of the eqn. (5.11) constitute a second order linear equation. To investigate the stability of eqn. (5.11), consider first the linear part of the equation. A Lyapunov function of the linear part could be found by the following calculation.

Let P be a positive defined matrix, such that

$$A^T P + PA = -I \tag{5.12}$$

where A is the dynamic matrix of linear part of eqn. (5.11) such that

$$A = \begin{bmatrix} 0 & 1 \\ -\rho_{i,i+1} & -a \end{bmatrix}.$$

By solving eqn. 5.12, one has

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

where,

$$p_{11} = \frac{a^2 + \rho_{i,i+1}^2 + \rho_{i,i+1}}{2a\rho_{i,i+1}}, \qquad p_{12} = \frac{1}{2\rho_{i,i+1}}, \qquad p_{22} = \frac{1 + \rho_{i,i+1}}{2a\rho_{i,i+1}}$$

Therefore, a Lyapunov function for the linear part is:

$$V_1 = (a^2 + \rho_{i,i+1}^2 + \rho_{i,i+1})x^2 + 2ax\dot{x} + (1 + \rho_{i,i+1})\dot{x}^2 \tag{5.13}$$

Now, consider the following positive definite function as a candidate Lyapunov function for the full equation of (5.11):

$$V = V_1 + \frac{2(1 + \rho_{i,i+1})}{3}\kappa'_{i,i+1}|x|x^2 \tag{5.14}$$

Substituting eqn. (5.11) into the derivative of eqn. (5.14) and, one has

$$\begin{aligned} \dot{V} &= -2a\rho_{i,i+1}x^2 - 2a\rho_{i,i+1}\dot{x}^2 - 2a\kappa'_{i,i+1}|x|x^2 \\ &\leq 0 \end{aligned}$$

Therefore, the system (5.7) is stable in the given region. It is known from above proof that a second order stable fuzzy gain system can always be designed by keeping the coefficients $\rho_{i,i+1}$ and $\kappa_{i,i+1}$ satisfying eqn. (5.10). For high order system, similar approach may be attempted.

5.6 Simulation Case Study

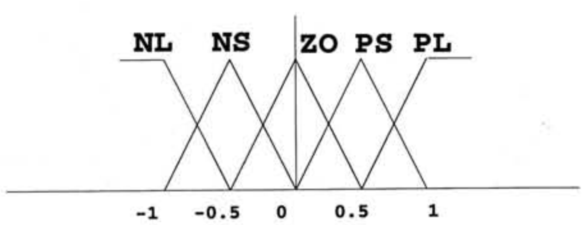
To illustrate the above discussion, a simulation case study is conducted with regulation problem on a second order linear system. Consider the following unstable system :

$$\ddot{x} = 5\dot{x} + 2r. \tag{5.15}$$

where, r is the input of the system. The performance measure, p, is defined as the position error x. Three state feedback gains, denoted by **SD** (small damping), **MD** (medium damping) and **LD** (large damping), are designed for large, medium, and small x. The values of x are characterized by the linguistic terms, **PL** (positive large), **PS** (positive small), **ZO** (about zero), **NS** (negative small) and **NL** (negative large). When x is large, **SD**, corresponds to a light damping system and is more efficient in bringing the system towards the set point. When x is small, however, **LD** is more efficient in damping out the dynamics and minimizing the overshoot. With such reasoning, a fuzzy gain inference system can be expressed as follows:

$$\text{If} \quad p \text{ is } P_i \ \Rightarrow \ c \text{ is } C_i$$

The damping factor, $\zeta = 0.5, 0.7, 0.9$ are selected for **SD**, **MD** and **LD**, respectively. The designed feedback gains are $3.5\dot{x} + 2x$, $3.5\dot{x} + x$ and $3.5\dot{x} + 0.62x$ for **SD**, **MD** and **LD**, respectively. The membership functions and fuzzy rule table are shown in Fig. 5.3. Steps response for different initial conditions, $x_o = 0.5, 0.8, 1.0, 1.5$, are shown in Fig. 5.4. It is seen that the fuzzy gain control system attains better performance as it has relative small over-shoot and fast rising time. Fig. 5.5 shows that the trajectories of the fuzzy system is bounded by that of purely small damping and large damping responses.

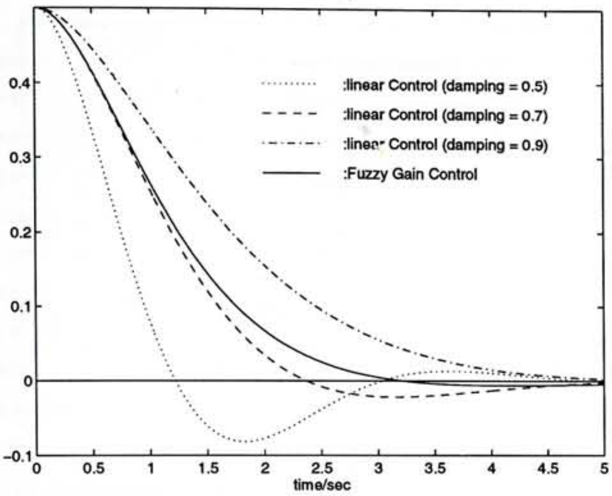


(a)

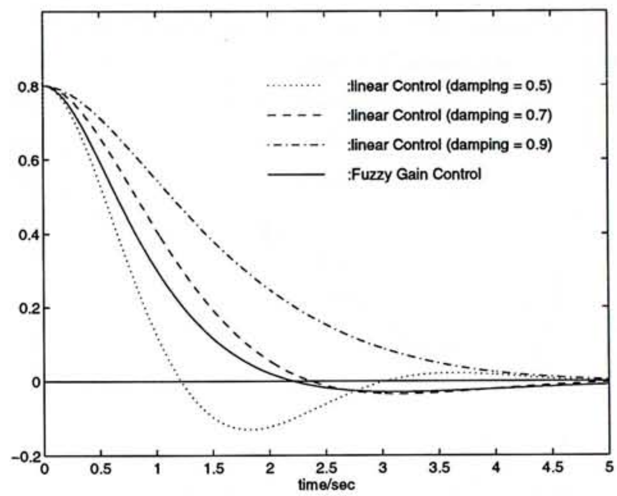
	p				
	NL	NS	ZO	PS	PL
C_i	SD	MD	LD	MD	SD

(b)

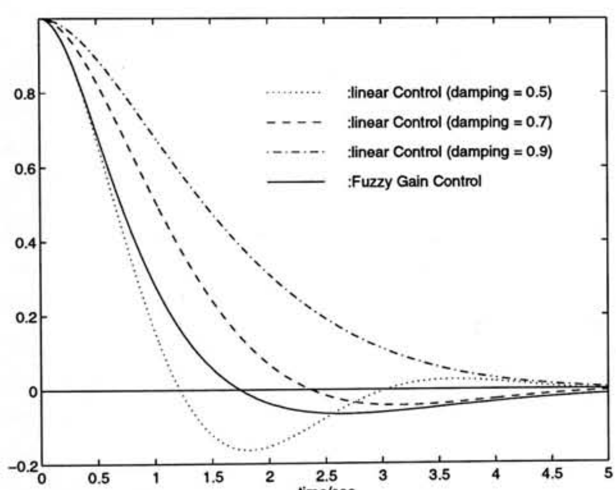
Fig 5.3 (a) Membership Functions; (b) Fuzzy Rule Table



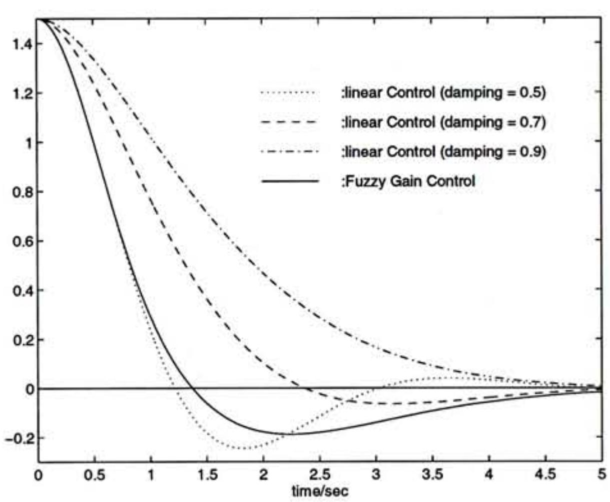
(a) initial condition $x_o = 0.5$



(b) initial condition $x_o = 0.8$

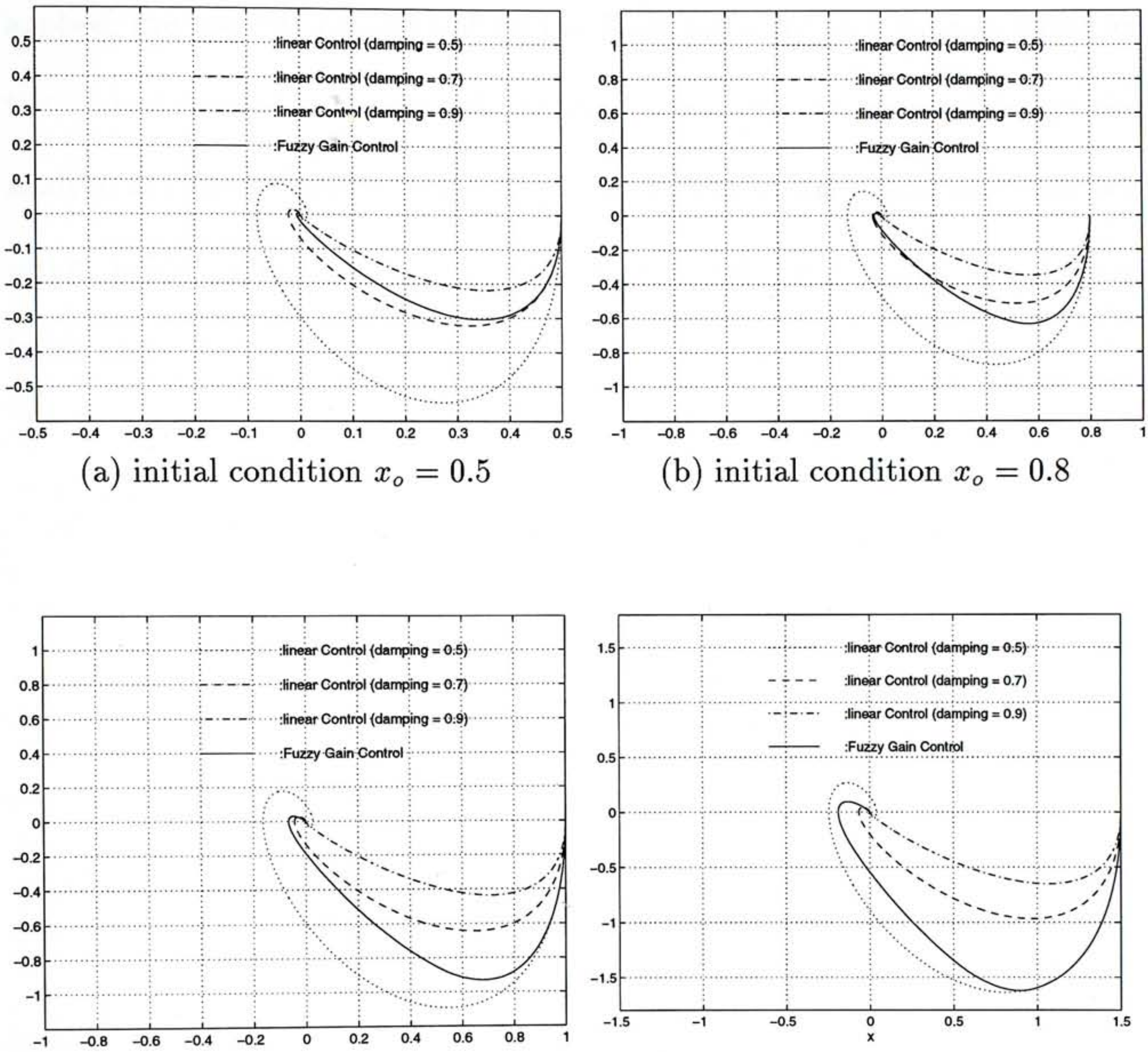


(c) initial condition $x_o = 1.0$



(d) initial condition $x_o = 1.5$

Fig 5.4 Response of Fuzzy Gain Control Vs Linear Control with different initial conditions



(a) initial condition $x_o = 0.5$

(b) initial condition $x_o = 0.8$

(c) initial condition $x_o = 1.0$

(d) initial condition $x_o = 1.5$

Fig 5.5 Flow Lines of Fuzzy Gain Control Vs Linear Control

5.7 Conclusion

While conventional fuzzy control inference produces the control action to be applied, the present method utilizes the fuzzy inference to infer gain value for the control action. This approach is more in-line with conventional design and analysis of controller using input/output gains and transfer functions, and as a result, would yield more tractable performance and stability analysis. A simulation case study demonstrates better performance with the present approach.

Chapter 6

Identification/Control Design Example

To illustrate the proposed idea from previous sections, a design example involving fuzzy identification and control is shown here. Consider the cart-inverted pendulum system shown in Fig. 6.1, where θ is the angle between the pole and the vertical line; m is mass of the pole; l is the half length of the pole; M is the mass of the cart; f is the force applied to the cart; and x is the position of the cart. Assuming that there is no friction in the system, the dynamic equations of the system can be written as the following nonlinear differential equations [44],

$$\ddot{\theta} = \frac{g\,sin\theta + cos\theta\,\left(\frac{-f - m\,l\,\dot{\theta}^2\,sin\theta}{M+m}\right)}{l\left(\frac{4}{3} - \frac{m\,cos^2\theta}{M+m}\right)} \tag{6.1}$$

$$\ddot{x} = \frac{f + m\, l(\dot{\theta}^2\, sin\theta - \ddot{\theta}\, cos\theta)}{M + m} \tag{6.2}$$

where,

$$\begin{aligned} M &= 0.455 \quad \text{kg},\\ m &= 0.210 \quad \text{kg},\\ l &= 0.305 \quad \text{m},\\ \text{and } g &= 9.800 \quad \text{m}/s^2. \end{aligned}$$

If we consider $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x$ and $x_4 = \dot{x}$, eqn. (6.1) and eqn. (6.2) can be written as

$$\dot{x_1} = x_2 \tag{6.3}$$

$$\dot{x_2} = \frac{gsinx_1 + cosx_1\,(\frac{-f-m\,l\,x_2^2\,sinx_1}{M+m})}{l(\frac{4}{3} - \frac{m\,cos^2 x_1}{M+m})} \tag{6.4}$$

$$\dot{x_3} = x_4 \tag{6.5}$$

$$\dot{x_4} = \frac{f + m\, l(x_2^2\, sinx_1 - \dot{x_2}\, cosx_1)}{M + m} \tag{6.6}$$

Fuzzy identification is conducted using the angular position and velocity, θ and $\dot{\theta}$, as fuzzy variables. Eqn. (6.3) to (6.6) are excited with 2000 random input after which fuzzy regional system identification with TSK model is conducted

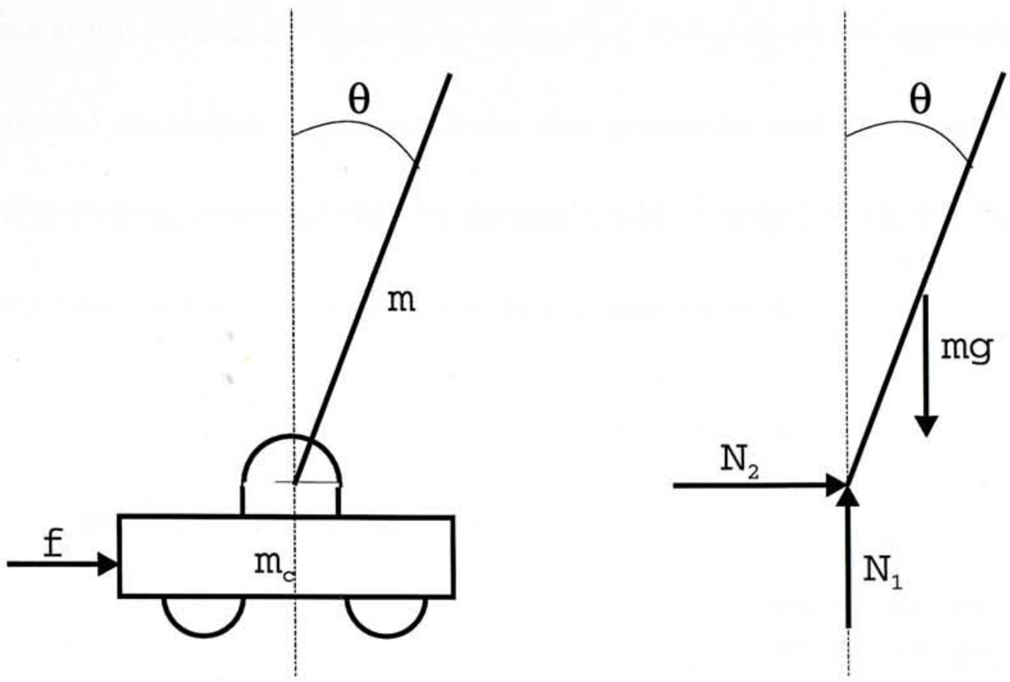
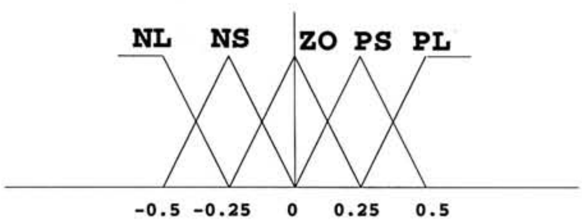


Fig 6.1 The Cart-Inverted Pendulum System

adopting membership functions shown in Fig. 6.2(a). The identification results are tabulated in Table 6.1 and Table 6.2.

Fuzzy gain controllers are then designed based on the TSK models obtained in Table 6.1 and Table 6.2. The performance measure, p, is defined to be θ. Three state feedback gains, denoted by SD (small damping), MD (medium damping) and LD (large damping), are designed with damping factors, $\zeta = 0.5, 0.7, 0.9$, respectively. The fuzzy rule table for the system is shown in Fig. 6.2(b). The designed feedback gains are tabulated in Table 6.3. The response of the fuzzy gain control system with initial condition $\theta_o = 0.5$ is shown in Fig. 6.3. It is seen that the performance of the fuzzy gain controller attains better performance as it has relative small over-shoot and fast rising time. Decomposable approximations of the designed controller in Table 6.3 are also generated and are shown in Table 6.4. Corresponding response due to decomposable control is shown in Fig 6.4 and is very close to that of the original fuzzy gain control.



(a)

	P				
	NL	NS	ZO	PS	PL
C_i	SD	MD	LD	MD	SD

(b)

Fig 6.2 (a) Membership Functions; (b) Fuzzy Rule Table

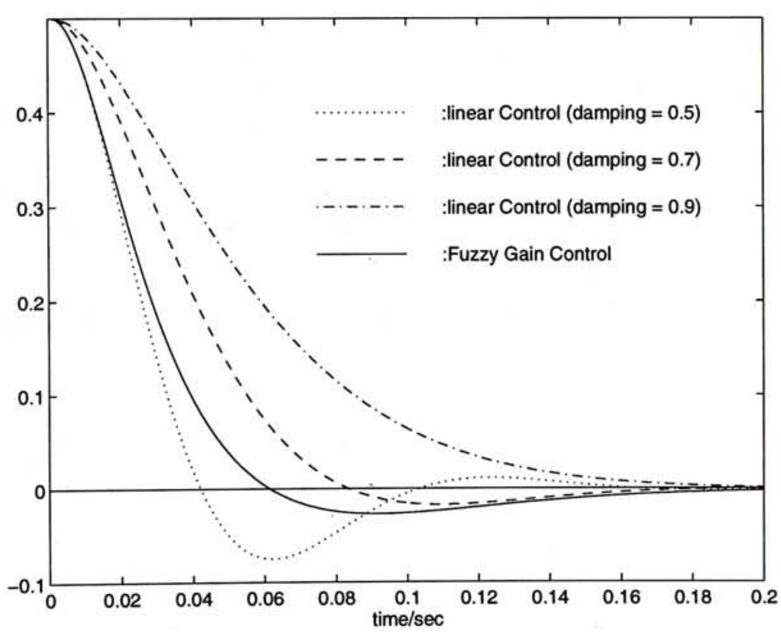


Fig 6.3 Response of Fuzzy Gain Control Vs Linear System for $x_1 = 0.5$

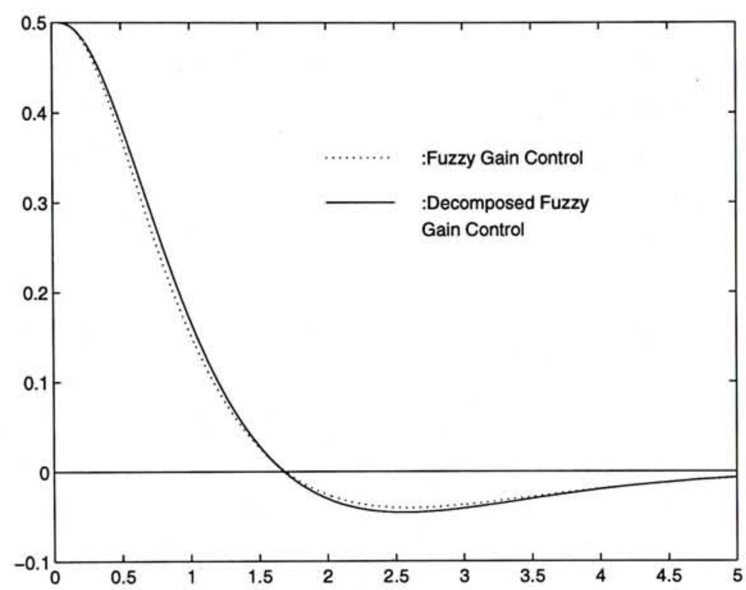


Fig 6.4 Response of Fuzzy Gain Control Vs Its Decomposed Approximation

		x_1				
		NL	NS	ZO	PS	PL
x_2	NL	$26.866\ x_1$ $+1.691\ x_2$ $-0.019\ x_3$ $+0.020\ x_4$ $-4.013\ f$	$28.198\ x_1$ $+0.752\ x_2$ $-0.021\ x_3$ $-0.001\ x_4$ $-4.055\ f$	$28.631\ x_1$ $+0.003\ x_2$ $-0.027\ x_3$ $-0.024\ x_4$ $-4.056\ f$	$28.224\ x_1$ $-0.732\ x_2$ $-0.008\ x_3$ $+0.038\ x_4$ $-4.043\ f$	$26.490\ x_1$ $-2.043\ x_2$ $-0.024\ x_3$ $-0.003\ x_4$ $-4.027\ f$
	NS	$29.237\ x_1$ $+0.715\ x_2$ $+0.008\ x_3$ $-0.021\ x_4$ $-4.635\ f$	$30.220\ x_1$ $+0.400\ x_2$ $+0.012\ x_3$ $+0.002\ x_4$ $-4.595\ f$	$30.657\ x_1$ $+0.004\ x_2$ $+0.004\ x_3$ $+0.016\ x_4$ $-4.628\ f$	$30.199\ x_1$ $-0.435\ x_2$ $+0.021\ x_3$ $-0.003\ x_4$ $-4.612\ f$	$29.192\ x_1$ $-0.737\ x_2$ $+0.005\ x_3$ $+0.004\ x_4$ $-4.635\ f$
	ZO	$30.823\ x_1$ $+0.002\ x_2$ $-0.006\ x_3$ $+0.006\ x_4$ $-4.867\ f$	$31.727\ x_1$ $-0.004\ x_2$ $-0.001\ x_3$ $+0.002\ x_4$ $-4.882\ f$	$32.158\ x_1$ $-0.015\ x_2$ $-0.005\ x_3$ $-0.015\ x_4$ $-4.875\ f$	$31.694\ x_1$ $-0.002\ x_2$ $-0.001\ x_3$ $+0.002\ x_4$ $-4.874\ f$	$30.818\ x_1$ $+0.004\ x_2$ $+0.001\ x_3$ $-0.003\ x_4$ $-4.878\ f$
	PS	$29.169\ x_1$ $-0.732\ x_2$ $-0.007\ x_3$ $-0.015\ x_4$ $-4.612\ f$	$30.282\ x_1$ $-0.369\ x_2$ $-0.002\ x_3$ $+0.010\ x_4$ $-4.629\ f$	$30.659\ x_1$ $-0.034\ x_2$ $-0.003\ x_3$ $+0.011\ x_4$ $-4.618\ f$	$30.196\ x_1$ $+0.446\ x_2$ $-0.002\ x_3$ $+0.014\ x_4$ $-4.624\ f$	$29.209\ x_1$ $+0.702\ x_2$ $-0.006\ x_3$ $+0.005\ x_4$ $-4.609\ f$
	PL	$26.756\ x_1$ $-1.836\ x_2$ $+0.001\ x_3$ $+0.019\ x_4$ $-4.051\ f$	$28.194\ x_1$ $-0.753\ x_2$ $-0.025\ x_3$ $-0.017\ x_4$ $-4.025\ f$	$28.613\ x_1$ $+0.142\ x_2$ $-0.004\ x_3$ $+0.010\ x_4$ $-4.031\ f$	$28.266\ x_1$ $+0.672\ x_2$ $+0.008\ x_3$ $-0.007\ x_4$ $-4.077\ f$	$26.815\ x_1$ $+1.633\ x_2$ $+0.025\ x_3$ $-0.009\ x_4$ $-4.019\ f$

Table 6.1: Fuzzy Output for $\dot{x_2}$

		x_1				
		NL	NS	ZO	PS	PL
		$-2.134\ x_1$	$-2.417\ x_1$	$-2.453\ x_1$	$-2.397\ x_1$	$-2.138\ x_1$
		$-0.306\ x_2$	$-0.076\ x_2$	$-0.012\ x_2$	$+0.103\ x_2$	$+0.295\ x_2$
	NL	$-0.009\ x_3$	$-0.003\ x_3$	$+0.004\ x_3$	$-0.006\ x_3$	$+0.005\ x_3$
		$-0.006\ x_4$	$+0.003\ x_4$	$+0.003\ x_4$	$-0.001\ x_4$	$+0.003\ x_4$
		$+1.856\ f$	$+1.853\ f$	$+1.855\ f$	$+1.851\ f$	$+1.847\ f$
		$-2.594\ x_1$	$-2.765\ x_1$	$-2.866\ x_1$	$-2.783\ x_1$	$-2.602\ x_1$
		$-0.123\ x_2$	$-0.086\ x_2$	$+0.002\ x_2$	$+0.070\ x_2$	$+0.120\ x_2$
	NS	$-0.001\ x_3$	$-0.000\ x_3$	$+0.004\ x_3$	$+0.000\ x_3$	$-0.002\ x_3$
		$-0.001\ x_4$	$+0.000\ x_4$	$-0.005\ x_4$	$+0.005\ x_4$	$-0.004\ x_4$
		$+1.936\ f$	$+1.934\ f$	$+1.935\ f$	$+1.938\ f$	$+1.937\ f$
		$-2.906\ x_1$	$-3.060\ x_1$	$-3.139\ x_1$	$-3.063\ x_1$	$-2.909\ x_1$
		$+0.001\ x_2$	$+0.000\ x_2$	$+0.004\ x_2$	$+0.001\ x_2$	$+0.001\ x_2$
x_2	ZO	$+0.001\ x_3$	$+0.001\ x_3$	$+0.000\ x_3$	$-0.000\ x_3$	$+0.001\ x_3$
		$+0.001\ x_4$	$+0.001\ x_4$	$-0.000\ x_4$	$+0.000\ x_4$	$-0.003\ x_4$
		$+1.975\ f$	$+1.975\ f$	$+1.975\ f$	$+1.976\ f$	$+1.975\ f$
		$-2.589\ x_1$	$-2.769\ x_1$	$-2.863\ x_1$	$-2.770\ x_1$	$-2.597\ x_1$
		$+0.125\ x_2$	$+0.082\ x_2$	$-0.011\ x_2$	$-0.080\ x_2$	$-0.121\ x_2$
	PS	$-0.002\ x_3$	$+0.002\ x_3$	$-0.001\ x_3$	$+0.003\ x_3$	$+0.002\ x_3$
		$-0.002\ x_4$	$+0.001\ x_4$	$+0.001\ x_4$	$-0.002\ x_4$	$+0.005\ x_4$
		$+1.935\ f$	$+1.936\ f$	$+1.935\ f$	$+1.934\ f$	$+1.938\ f$
		$-2.127\ x_1$	$-2.412\ x_1$	$-2.452\ x_1$	$-2.412\ x_1$	$-2.092\ x_1$
		$+0.306\ x_2$	$+0.092\ x_2$	$+0.037\ x_2$	$-0.091\ x_2$	$-0.314\ x_2$
	PL	$-0.003\ x_3$	$-0.003\ x_3$	$+0.009\ x_3$	$-0.017\ x_3$	$-0.001\ x_3$
		$+0.008\ x_4$	$-0.002\ x_4$	$-0.008\ x_4$	$+0.014\ x_4$	$-0.007\ x_4$
		$+1.844\ f$	$+1.853\ f$	$+1.852\ f$	$+1.849\ f$	$+1.856\ f$

Table 6.2: Fuzzy Output for $\dot{x_4}$

		x_1 NL	x_1 NS	x_1 ZO	x_1 PS	x_1 PL
x_2	NL	$280.47\ x_1$ $+36.88\ x_2$ $+86.70\ x_3$ $+45.42\ x_4$	$266.48\ x_1$ $+35.66\ x_2$ $+42.08\ x_3$ $+44.16\ x_4$	$270.58\ x_1$ $+35.05\ x_2$ $+25.43\ x_3$ $+43.22\ x_4$	$290.39\ x_1$ $+36.42\ x_2$ $+42.02\ x_3$ $+46.44\ x_4$	$337.99\ x_1$ $+40.73\ x_2$ $+89.11\ x_3$ $+56.34\ x_4$
	NS	$252.95\ x_1$ $+31.92\ x_2$ $+80.80\ x_3$ $+44.03\ x_4$	$242.19\ x_1$ $+31.13\ x_2$ $+39.48\ x_3$ $+41.69\ x_4$	$237.27\ x_1$ $+30.44\ x_2$ $+24.15\ x_3$ $+40.74\ x_4$	$249.66\ x_1$ $+31.24\ x_2$ $+39.62\ x_3$ $+42.58\ x_4$	$269.92\ x_1$ $+33.02\ x_2$ $+80.95\ x_3$ $+47.39\ x_4$
	ZO	$248.02\ x_1$ $+30.48\ x_2$ $+77.04\ x_3$ $+43.74\ x_4$	$231.20\ x_1$ $+29.00\ x_2$ $+37.73\ x_3$ $+40.28\ x_4$	$225.25\ x_1$ $+28.41\ x_2$ $+22.93\ x_3$ $+38.75\ x_4$	$231.51\ x_1$ $+29.05\ x_2$ $+37.73\ x_3$ $+40.28\ x_4$	$247.36\ x_1$ $+30.39\ x_2$ $+77.14\ x_3$ $+43.65\ x_4$
	PS	$270.53\ x_1$ $+33.05\ x_2$ $+80.78\ x_3$ $+47.12\ x_4$	$246.95\ x_1$ $+31.01\ x_2$ $+39.26\ x_3$ $+42.30\ x_4$	$238.94\ x_1$ $+30.53\ x_2$ $+23.97\ x_3$ $+40.84\ x_4$	$239.74\ x_1$ $+30.98\ x_2$ $+39.52\ x_3$ $+41.80\ x_4$	$254.93\ x_1$ $+32.30\ x_2$ $+80.65\ x_3$ $+44.45\ x_4$
	PL	$329.50\ x_1$ $+40.01\ x_2$ $+88.37\ x_3$ $+55.29\ x_4$	$291.56\ x_1$ $+36.32\ x_2$ $+41.92\ x_3$ $+45.82\ x_4$	$267.71\ x_1$ $+35.24\ x_2$ $+25.84\ x_3$ $+43.16\ x_4$	$267.53\ x_1$ $+35.60\ x_2$ $+42.10\ x_3$ $+44.62\ x_4$	$274.05\ x_1$ $+36.23\ x_2$ $+88.31\ x_3$ $+44.78\ x_4$

Table 6.3: Fuzzy Gain Control

x_1	NL	$299.86x_1 + 37.91x_2 + 85.90x_3 + 49.50x_4$
	NS	$279.24x_1 + 36.06x_2 + 43.26x_3 + 45.23x_4$
	ZO	$271.52x_1 + 35.37x_2 + 27.63x_3 + 43.72x_4$
	PS	$279.33x_1 + 36.10x_2 + 43.36x_3 + 45.52x_4$
	PL	$300.42x_1 + 37.97x_2 + 86.40x_3 + 49.70x_4$
x_2	NL	$3.11x_1 + 0.26x_2 - 0.24x_3 + 0.38x_4$
	NS	$-35.67x_1 - 5.13x_2 - 4.31x_3 - 3.44x_4$
	ZO	$-49.40x_1 - 7.22x_2 - 6.80x_3 - 5.40x_4$
	PS	$-35.85x_1 - 5.11x_2 - 4.47x_3 - 3.43x_4$
	PL	0

Table 6.4: Decomposed Fuzzy Gain Control

Chapter 7

Conclusion

This thesis serves to establish linkages between conventional control and fuzzy control techniques. Two area, namely, fuzzy system identification and controller design, have been tackled with some new results.

For system identification, application of linear regression formulation towards fuzzy identification is conducted. The problem of linearly dependent variable is pointed out. Fuzzy regional identification is also developed with fuzzy membership function acting as weighting function in a localized identification problem. The regional approach minimizes errors within individual domain and results in larger global error. However, the regional approach does not suffer from the problem of linearly dependent variables, as each subsystem handles input data individually. Furthermore, it is applicable even in the case that some regions are not sampled.

For controller design, performance based fuzzy gain controller is developed. The proposed approach utilizes a fuzzy system to produce the "gain" value of the control input and is more in line with the classical feedback techniques. As such, stability of the resultant system is more readily checked. An approach which demonstrates the stability of second-order system by Lyapunov method is presented here.

The present work also studies conditions underwhich a general fuzzy system can be decomposable into fuzzy subsystems of single variable. The decomposed system has the advantage of requiring a reduced number of fuzzy rules as compared to the original system. In case that the original fuzzy system does not meet decomposability conditions, a decomposition approximation can be obtained via minimization of a quadratic error cost functions.

Continuation of the present work can be carried out along several directions. They include, firstly, development of fuzzy adaptive controller based on parametric model. With the on-line fuzzy identification results developed here, the adaptive mechanism seems to be a natural extension. Secondly, a systematic procedure to determine an effective performance index for fuzzy gain controller should be studied for optimized performance. Thirdly, as the additive fuzzy rule decomposition is developed in this study, multiplicative fuzzy rule decomposition should also be investigated.

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